

Fitting CAR and SAR Models

Lecture 19

Dr. Colin Rundel

Fitting areal models

Revised CAR Model

- Conditional Model

$$y(s_i) | \mathbf{y}_{-s_i} \sim N \left(X_{i\cdot} \beta + \phi \sum_{j=1}^n \frac{A_{ij}}{D_{ii}} (y(s_j) - X_{j\cdot} \beta), \sigma^2 D_{ii}^{-1} \right)$$

- Joint Model

$$\mathbf{y} \sim N(\mathbf{X}\beta, \sigma^2(\mathbf{D} - \phi\mathbf{A})^{-1})$$

SAR - lag model

Let us consider what happens to our derivation of the SAR model when we include a $\mathbf{X}\beta$ in our formula model.

$$\mathbf{y} = \mathbf{X}\beta + \phi \mathbf{D}^{-1} \mathbf{A} \mathbf{y} + \epsilon$$
$$\epsilon \sim N(\mathbf{0}, \sigma^2 \mathbf{D}^{-1})$$

$$\mathbf{y} = \mathbf{X}\beta + \phi \mathbf{D}^{-1} \mathbf{A} \mathbf{y} + \epsilon$$

$$\mathbf{y} - \phi \mathbf{D}^{-1} \mathbf{A} \mathbf{y} = \mathbf{X}\beta + \epsilon$$

$$(\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{A}) \mathbf{y} = \mathbf{X}\beta + \epsilon$$

$$\mathbf{y} = (\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{A})^{-1} \mathbf{X}\beta + (\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{A})^{-1} \epsilon$$

Properties

$$E(\mathbf{y}) = (I - \phi \mathbf{D}^{-1} \mathbf{A})^{-1} \mathbf{X} \boldsymbol{\beta}$$

$$\begin{aligned} Var(\mathbf{y}) &= ((I - \phi \mathbf{D}^{-1} \mathbf{A})^{-1}) \sigma^2 \mathbf{D}^{-1} ((I - \phi \mathbf{D}^{-1} \mathbf{A})^{-1})^t \\ &= \sigma^2 ((I - \phi \mathbf{D}^{-1} \mathbf{A})^{-1}) \mathbf{D}^{-1} ((I - \phi \mathbf{D}^{-1} \mathbf{A})^{-1})^t \end{aligned}$$

This is the same behavior we saw with the AR(1) model, where the mean of the process is $\delta/(1 - \phi)$ and **not** δ .

This is a relatively minor issue but it does have practical implications when it comes to interpretation of the model parameters (particularly the slope coefficients $\boldsymbol{\beta}$).

SAR - error model

The previous model is sometimes referred to as the **SAR lag model**, a variation of this model, common in spatial econometrics, is the **SAR error model** which is defined as follows,

$$\begin{aligned}y &= \mathbf{X}\boldsymbol{\beta} + \mathbf{u} \\ \mathbf{u} &= \phi \mathbf{D}^{-1} \mathbf{A} \mathbf{u} + \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon} &\sim N(\mathbf{0}, \sigma^2 \mathbf{D}^{-1})\end{aligned}$$

As with the SAR lag model we can solve for $\mathbf{u}$,

$$\begin{aligned}\mathbf{u} &= \phi \mathbf{D}^{-1} \mathbf{A} \mathbf{u} + \boldsymbol{\epsilon} \\ \mathbf{u} - \phi \mathbf{D}^{-1} \mathbf{A} \mathbf{u} &= \boldsymbol{\epsilon} \\ \mathbf{u} &= (\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{A})^{-1} \boldsymbol{\epsilon}\end{aligned}$$

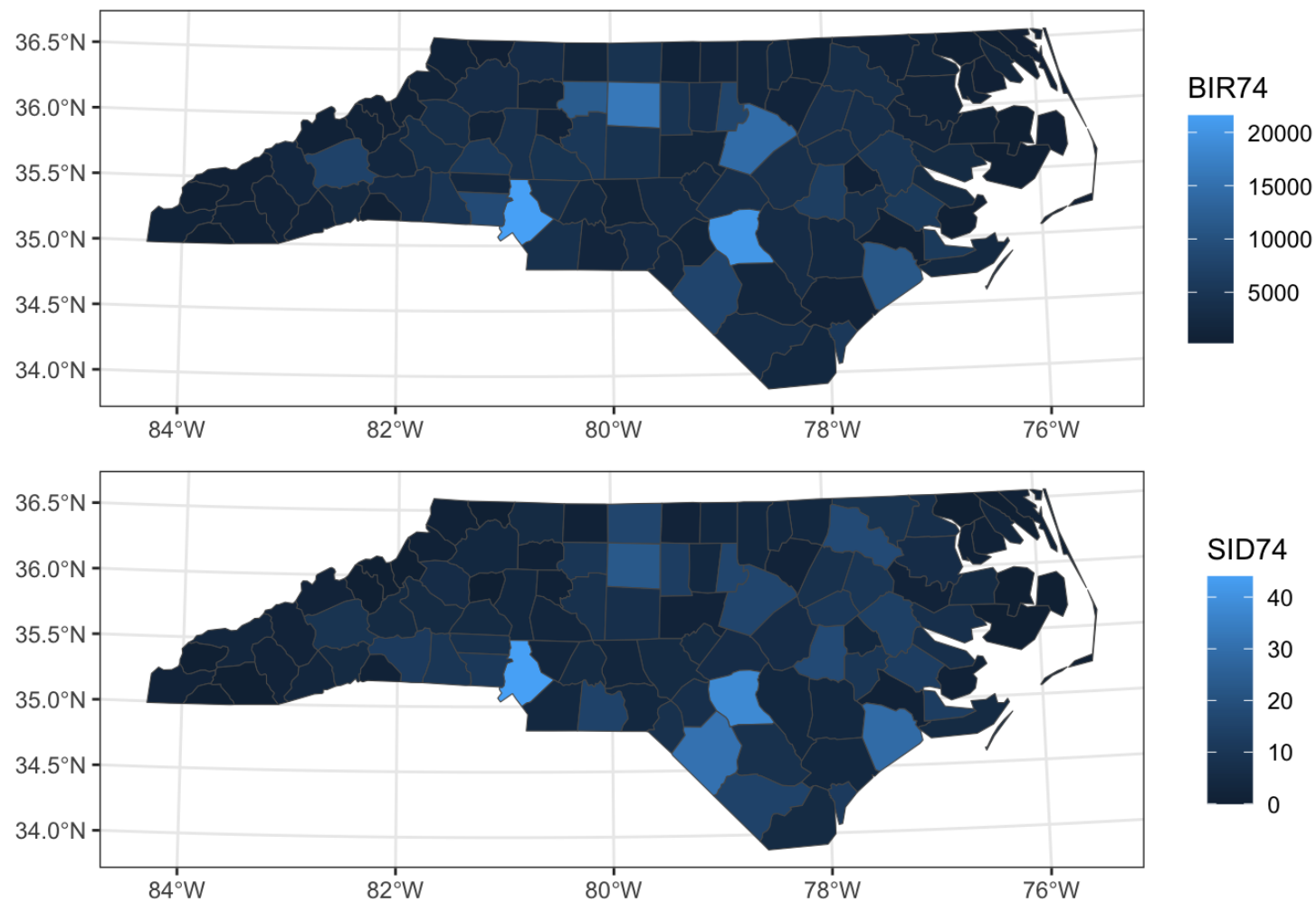
Properties

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + (\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{A})^{-1} \boldsymbol{\epsilon}$$

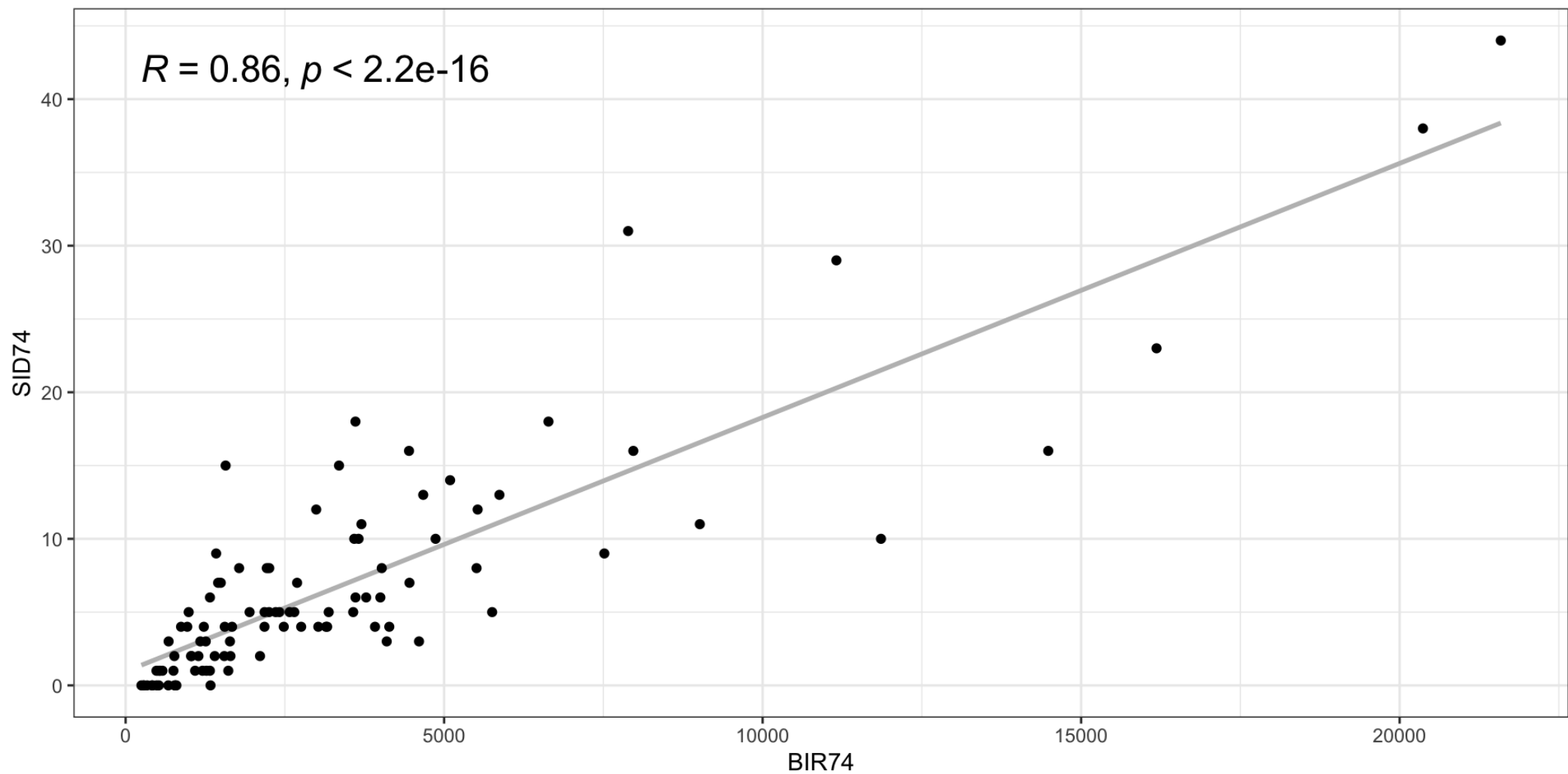
$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$$

$$Var(\mathbf{y}) = \sigma^2 \left((\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{A})^{-1} \right) \mathbf{D}^{-1} \left((\mathbf{I} - \phi \mathbf{D}^{-1} \mathbf{A})^{-1} \right)^t$$

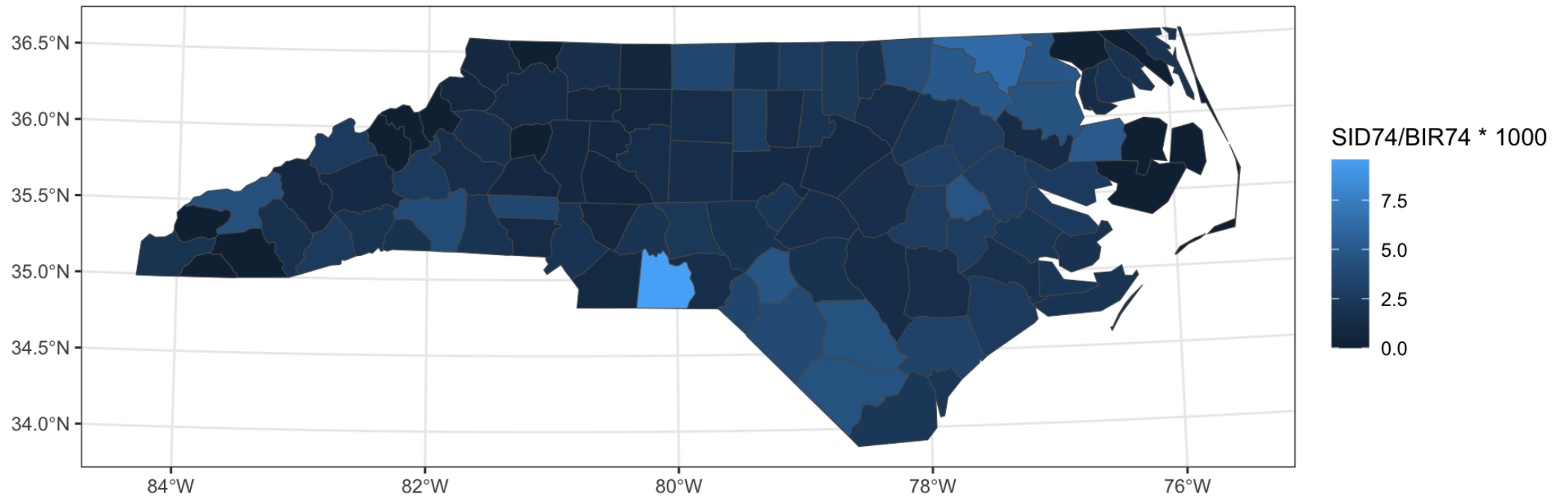
Example - NC SIDS



BIR74 VS SID74



```
1 ggplot() + geom_sf(data=nc, aes(fill=SID74/BIR74*1000))
```



Using spdep + spatialreg

```
1 A = st_touches(nc, sparse=FALSE)
2 (listW = spdep::mat2listw(A))
```

Characteristics of weights list object:

Neighbour list object:

Number of regions: 100

Number of nonzero links: 490

Percentage nonzero weights: 4.9

Average number of links: 4.9

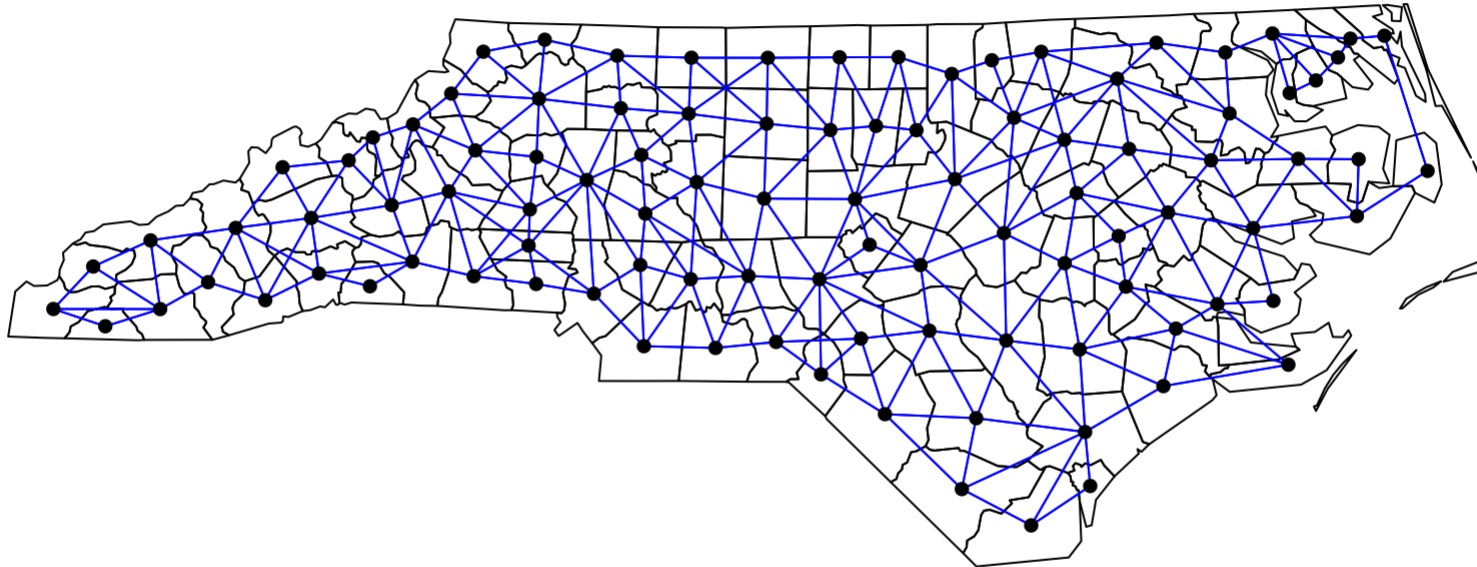
Weights style: M

Weights constants summary:

	n	nn	S0	S1	S2
M	100	10000	490	980	10696

Plotting listw

```
1 nc_coords = nc |> st_centroid() |> st_coordinates()  
2 plot(st_geometry(nc))  
3 plot(listW, nc_coords, add=TRUE, col="blue", pch=16)
```



Moran's I

```
1 spdep::moran.test(  
2   nc$SID74, listW  
3 )
```

Moran I test under randomisation

data: nc\$SID74
weights: listW

Moran I statistic standard deviate = 2.1707,
p-value = 0.01498
alternative hypothesis: greater

sample estimates:

Moran I statistic	Expectation
0.119089049	-0.010101010
Variance	
0.003542176	

```
1 spdep::moran.test(  
2   1000*nc$SID74/nc$BIR74, listW  
3 )
```

Moran I test under randomisation

data: 1000 * nc\$SID74/nc\$BIR74
weights: listW

Moran I statistic standard deviate = 3.6355,
p-value = 0.0001387
alternative hypothesis: greater

sample estimates:

Moran I statistic	Expectation
0.210046454	-0.010101010
Variance	
0.003666802	

Geary's C

```
1 spdep::geary.test(  
2   nc$SID74, listW  
3 )
```

Geary C test under randomisation

data: nc\$SID74
weights: listW

Geary C statistic standard deviate =
0.91949, p-value = 0.1789
alternative hypothesis: Expectation greater than
statistic

sample estimates:

Geary C statistic	Expectation
0.88988684	1.00000000
Variance	
0.01434105	

```
1 spdep::geary.test(  
2   1000*nc$SID74/nc$BIR74, listW  
3 )
```

Geary C test under randomisation

data: 1000 * nc\$SID74/nc\$BIR74
weights: listW

Geary C statistic standard deviate = 3.0989,
p-value = 0.0009711
alternative hypothesis: Expectation greater than
statistic

sample estimates:

Geary C statistic	Expectation
0.67796679	1.00000000
Variance	
0.01079878	

CAR Model

```
1 nc_car = spatialreg::spautolm(  
2   formula = 1000*SID74/BIR74 ~ 1, data = nc,  
3   listw = listW, family = "CAR"  
4 )  
5 summary(nc_car)
```

```
Call: spatialreg::spautolm(formula = 1000 * SID74/BIR74 ~ 1, data = nc,  
  listw = listW, family = "CAR")
```

Residuals:

Min	1Q	Median	3Q	Max
-2.13872	-0.83535	-0.22355	0.55014	7.68640

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	2.00203	0.24272	8.2484	2.22e-16

Lambda: 0.13062 LR test value: 8.6251 p-value: 0.0033157

Numerical Hessian standard error of lambda: 0.030475

Log likelihood: -182.3989

ML residual variance (sigma squared): 2.1205, (sigma: 1.4562)

Number of observations: 100

SAR Model (error)

```
1 nc_sar_err = spatialreg::spautolm(  
2   formula = 1000*SID74/BIR74 ~ 1, data = nc,  
3   listw = listW, family = "SAR"  
4 )  
5 summary(nc_sar_err)
```

```
Call: spatialreg::spautolm(formula = 1000 * SID74/BIR74 ~ 1, data = nc,  
  listw = listW, family = "SAR")
```

Residuals:

Min	1Q	Median	3Q	Max
-2.09307	-0.87039	-0.20274	0.51156	7.62830

Coefficients:

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	2.01084	0.23622	8.5127	< 2.2e-16

Lambda: 0.079934 LR test value: 8.8911 p-value: 0.0028657

Numerical Hessian standard error of lambda: 0.024597

Log likelihood: -182.2659

ML residual variance (sigma squared): 2.1622, (sigma: 1.4704)

Number of observations: 100

SAR Model (lag)

```
1 nc_sar_lag = spatialreg::lagsarlm(  
2   formula = 1000*SID74/BIR74 ~ 1, data = nc,  
3   listw = listW  
4 )  
5 summary(nc_sar_lag)
```

```
Call:spatialreg::lagsarlm(formula = 1000 * SID74/BIR74 ~ 1, data = nc,  
    listw = listW)
```

Residuals:

Min	1Q	Median	3Q	Max
-2.03968	-1.10398	-0.14413	0.54949	7.70889

Type: lag

Coefficients: (asymptotic standard errors)

	Estimate	Std. Error	z value	Pr(> z)
(Intercept)	1.40320	0.27296	5.1406	2.738e-07

Rho: 0.063331, LR test value: 7.5512, p-value: 0.0059971

Asymptotic standard error: 0.021655

z-value: 2.9246, p-value: 0.0034493

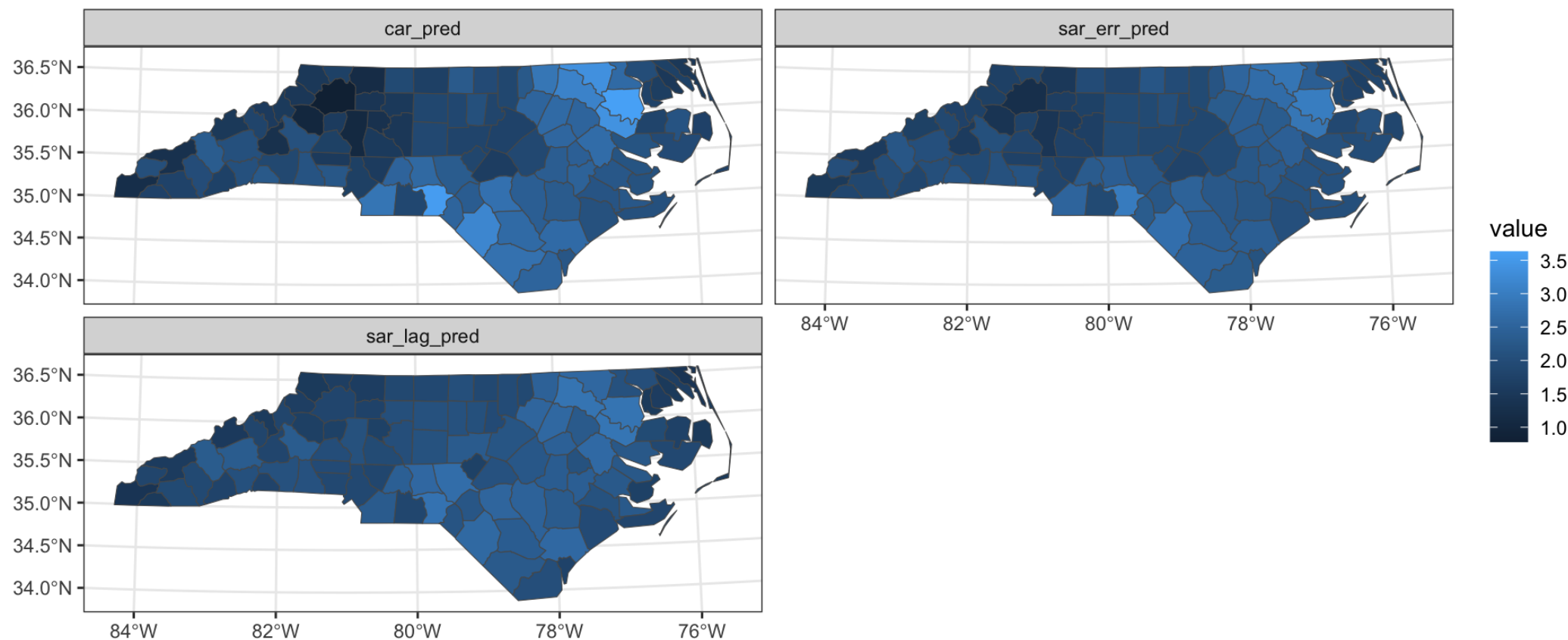
Wald statistic: 8.5531, p-value: 0.0034493

Log likelihood: -182.9358 for lag model Sta 344/644 - Fall 2023

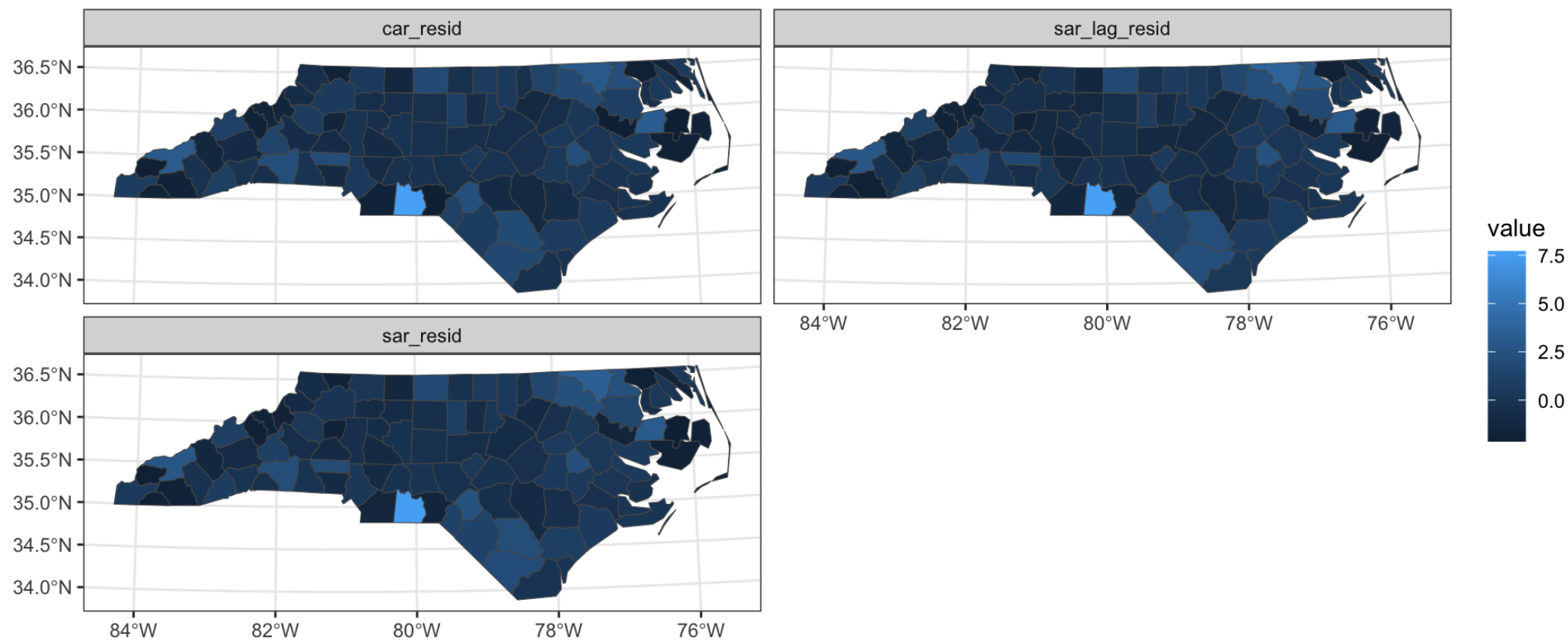
ML residual variance (sigma squared): 2.2232, (sigma: 1.491)

Number of observations: 100

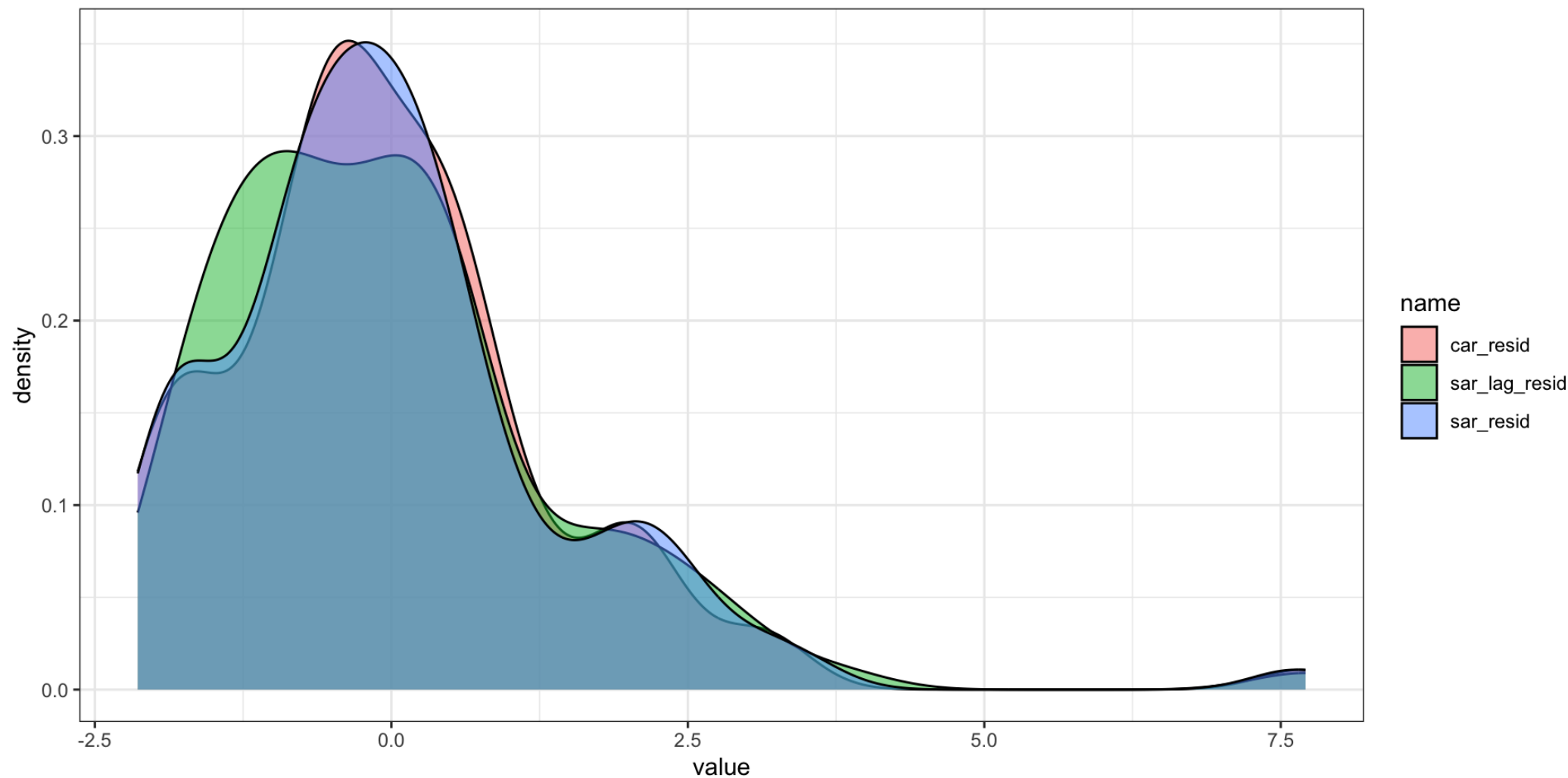
Predictions



Residuals



Residual distributions



Residual autocorrelation

```
1 spdep::moran.test(  
2   residuals(nc_car), listW,  
3   alternative = "two.sided"  
4 )
```

Moran I test under randomisation

```
data:  residuals(nc_car)  
weights: listW
```

```
Moran I statistic standard deviate = -1.7952, p-  
value = 0.07261
```

```
alternative hypothesis: two.sided
```

```
sample estimates:
```

Moran I statistic	Expectation
-------------------	-------------

Variance

-0.117449312	-0.010101010
0.003575538	

```
1 spdep::moran.test(  
2   residuals(nc_sar_err), listW,  
3   alternative = "two.sided"  
4 )
```

Moran I test under randomisation

```
data:  residuals(nc_sar_err)  
weights: listW
```

```
Moran I statistic standard deviate = 0.17958, p-  
value = 0.8575
```

```
alternative hypothesis: two.sided
```

```
sample estimates:
```

Moran I statistic	Expectation
-------------------	-------------

Variance

0.0006769267	-0.0101010101
0.0036020941	

```
1 spdep::moran.test(  
2   residuals(nc_sar_lag), listW,  
3   alternative = "two.sided"  
4 )
```

Moran I test under randomisation

```
data:  residuals(nc_sar_lag)  
weights: listW
```

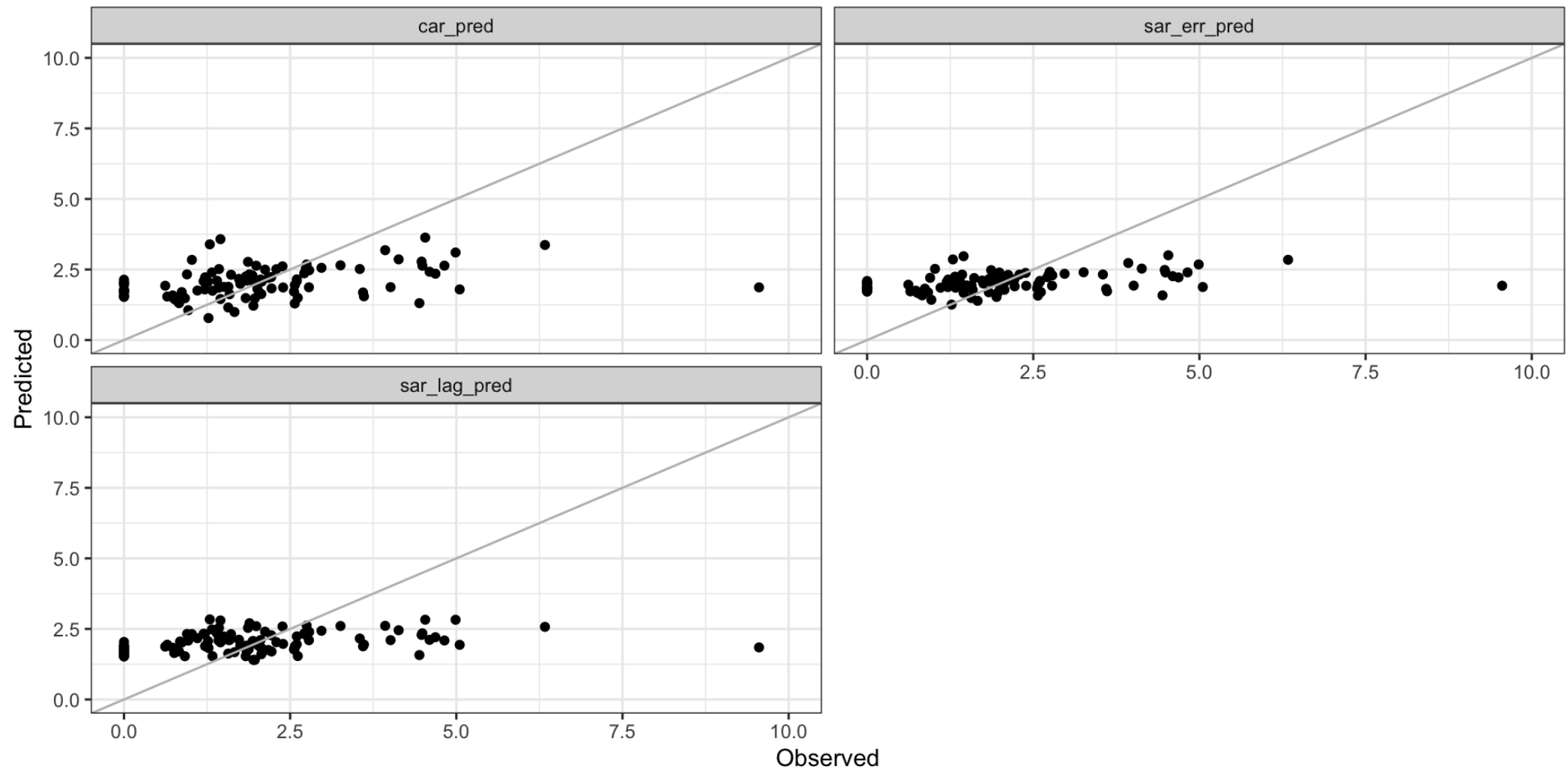
```
Moran I statistic standard deviate = 0.88877, p-  
value = 0.3741
```

```
alternative hypothesis: two.sided
```

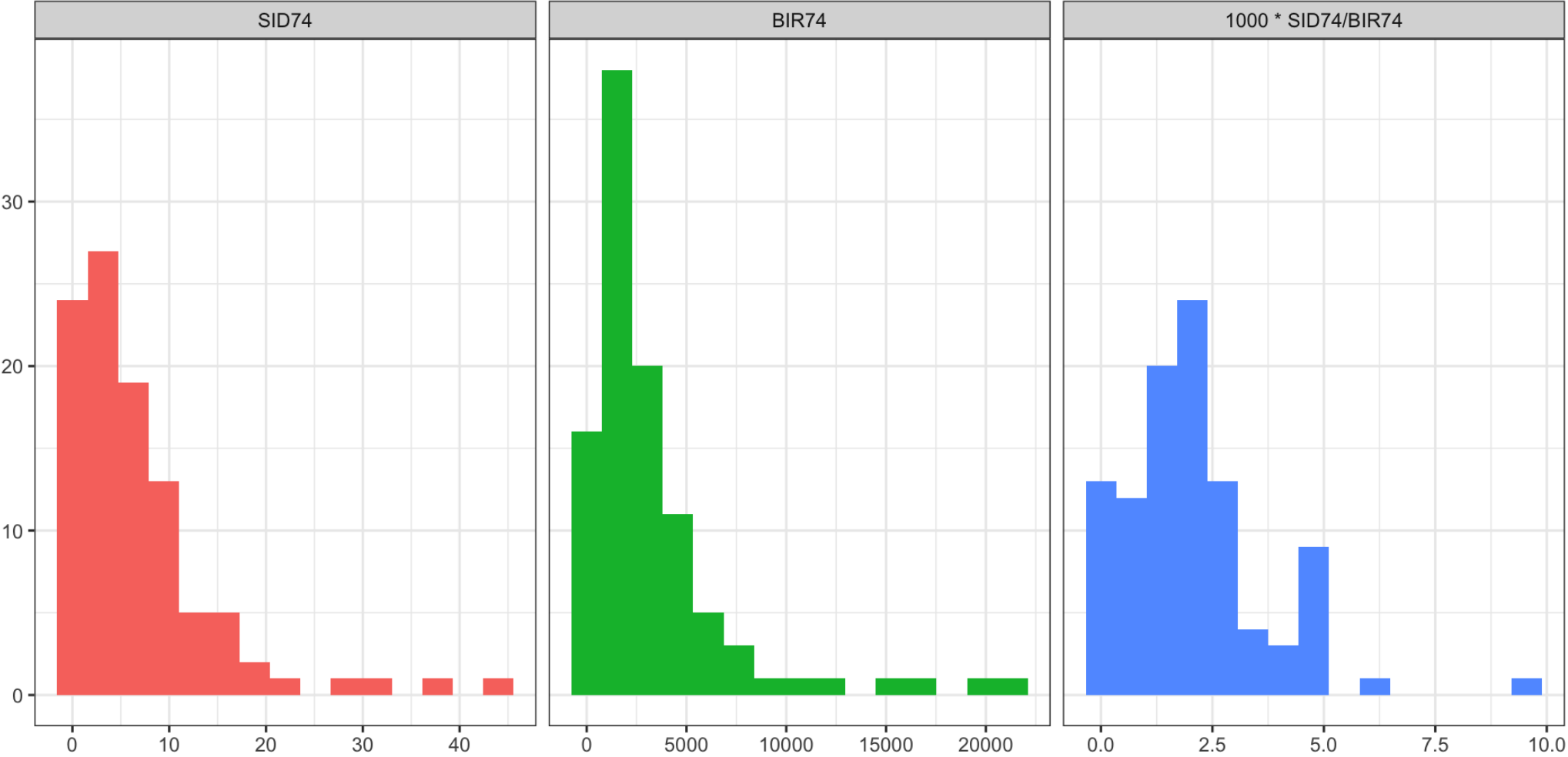
```
sample estimates:
```

Moran I statistic	Expectation
0.043255233	-0.010101010
0.003604055	

Observed vs Predicted



What's wrong?

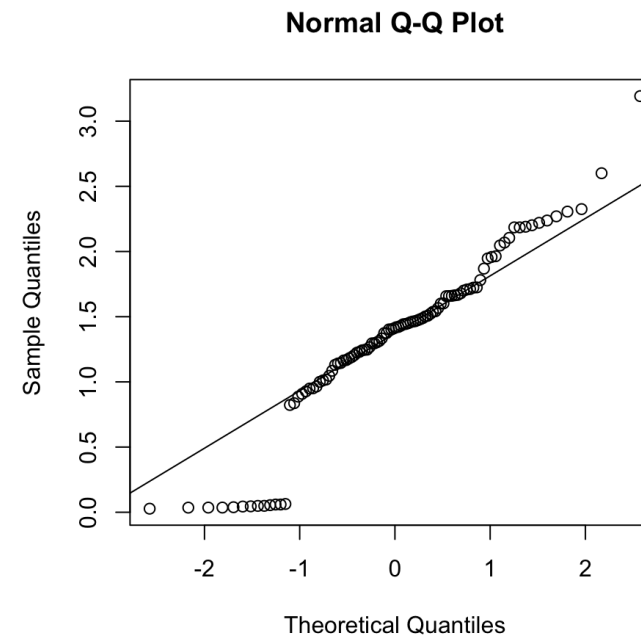
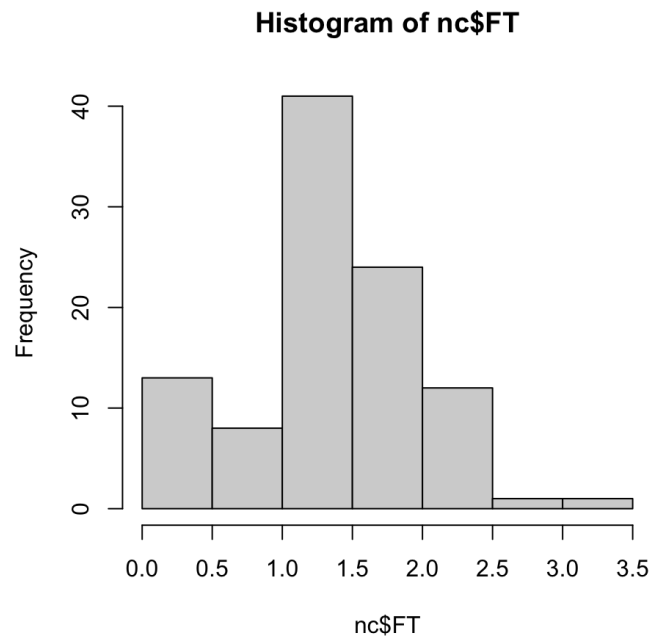


Transforming the data

Freeman-Tukey's transformation

This is the transformation used by Cressie and Road in Spatial Data Analysis of Regional Counts (1989).

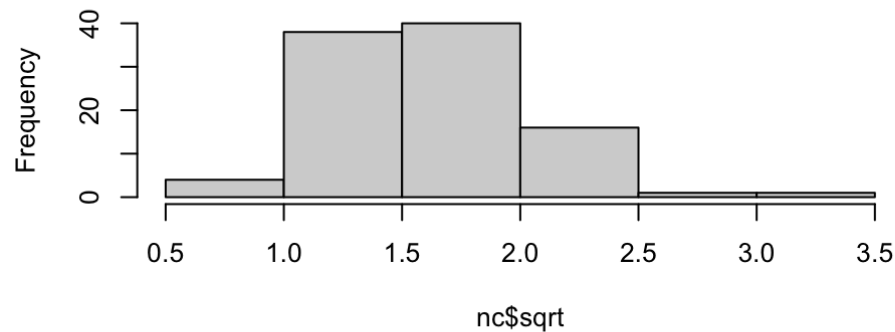
$$FT = \sqrt{1000} \left(\sqrt{\frac{SID74}{BIR74}} + \sqrt{\frac{SID74 + 1}{BIR74}} \right)$$



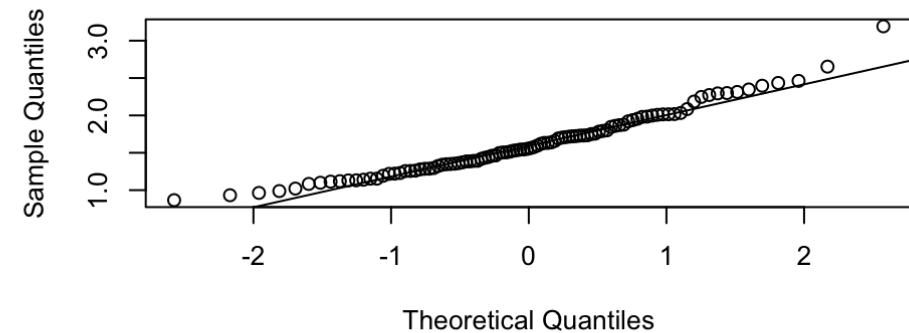
Other possibilities

```
1 nc = mutate(nc,  
2   sqrt = sqrt(1000*(SID74+1)/BIR74),  
3   log   = log(1000*(SID74+1)/BIR74),  
4 )
```

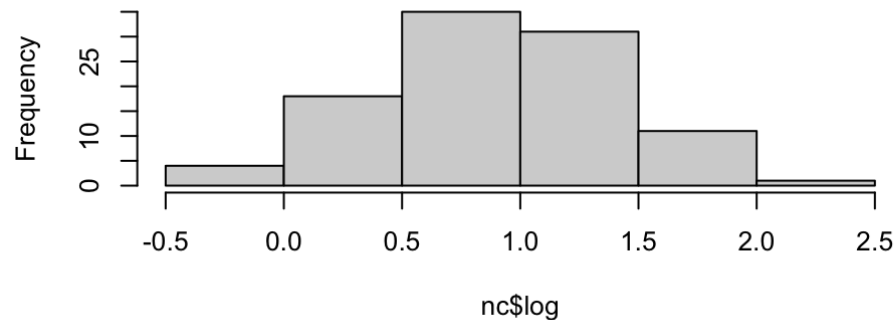
Histogram of nc\$sqrt



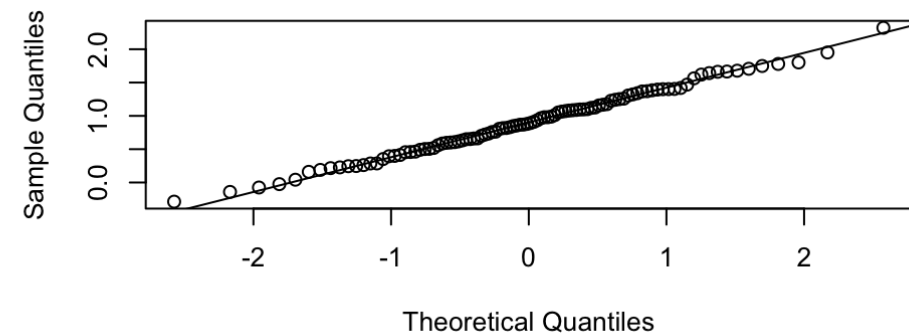
Normal Q-Q Plot



Histogram of nc\$log

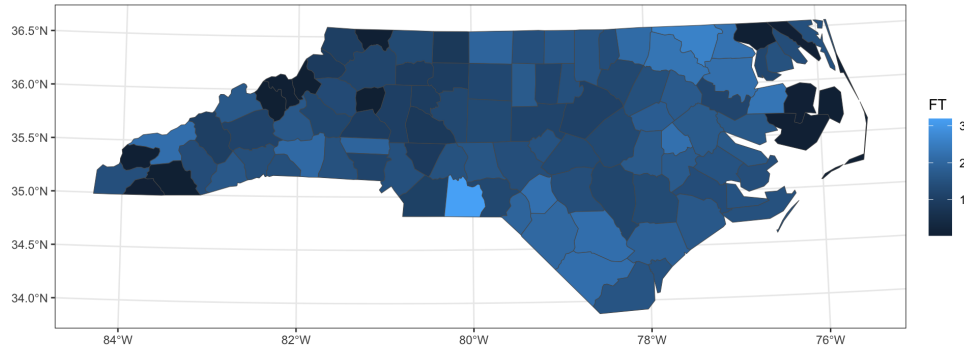


Normal Q-Q Plot



FT transformation

```
1 ggplot(nc) + geom_sf(aes(fill=FT))
```



```
1 spdep::moran.test(nc$FT, listW)
```

Moran I test under randomisation

data: nc\$FT

weights: listW

Moran I statistic standard deviate = 3.664, p-value = 0.0001242

alternative hypothesis: greater

sample estimates:

Moran I statistic	Expectation
-------------------	-------------

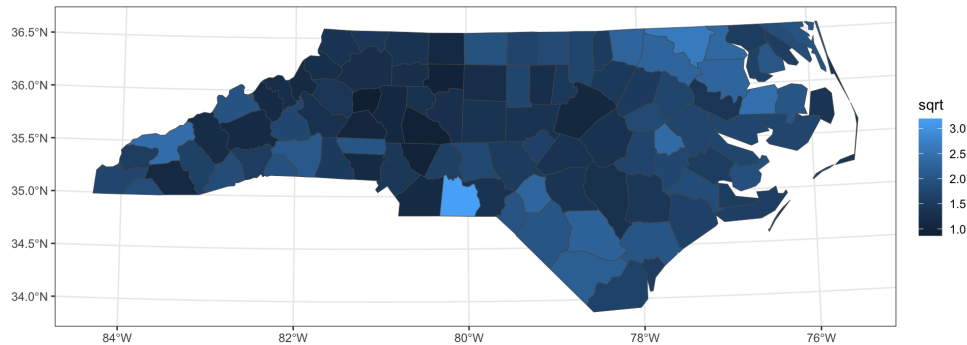
Variance	
----------	--

0.216246481	-0.010101010
-------------	--------------

0.003816298	
-------------	--

sqrt transformation

```
1 ggplot(nc) + geom_sf(aes(fill=sqrt))
```



```
1 spdep::moran.test(nc$sqrt, listW)
```

Moran I test under randomisation

data: nc\$sqrt

weights: listW

Moran I statistic standard deviate = 4.5217, p-value = 3.067e-06

alternative hypothesis: greater

sample estimates:

Moran I statistic	Expectation
-------------------	-------------

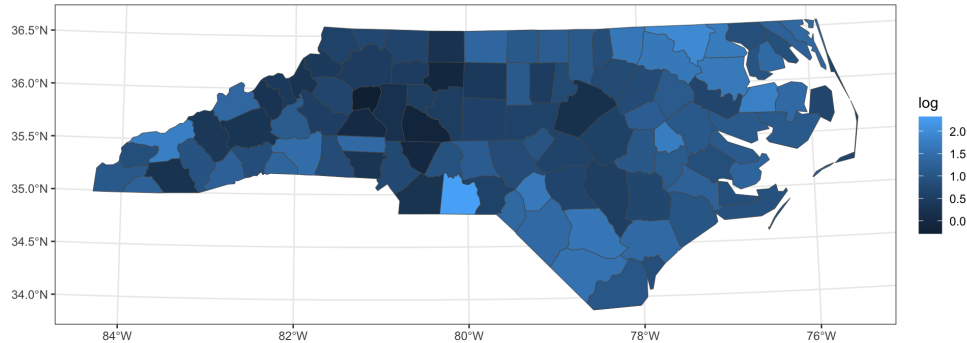
Variance	
----------	--

0.268600322	-0.010101010
-------------	--------------

0.003798988	
-------------	--

log transformation

```
1 ggplot(nc) + geom_sf(aes(fill=log))
```



```
1 spdep::moran.test(nc$log, listW)
```

Moran I test under randomisation

data: nc\$log

weights: listW

Moran I statistic standard deviate = 4.9895, p-value = 3.027e-07

alternative hypothesis: greater

sample estimates:

Moran I statistic	Expectation
-------------------	-------------

Variance	
----------	--

0.299245438	-0.010101010
-------------	--------------

0.003843927	
-------------	--

Models

CAR

```
1 nc_car_ft    = spatialreg::spautolm(formula = FT ~ 1,    data = nc, listw = listW, family = "CAR")
2 nc_car_sqrt  = spatialreg::spautolm(formula = sqrt ~ 1, data = nc, listw = listW, family = "CAR")
3 nc_car_log   = spatialreg::spautolm(formula = log ~ 1,   data = nc, listw = listW, family = "CAR")
```

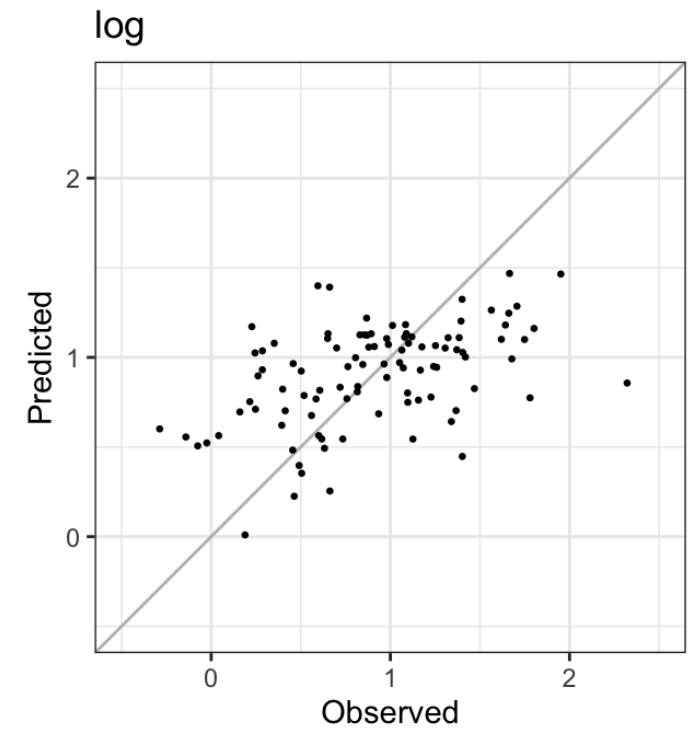
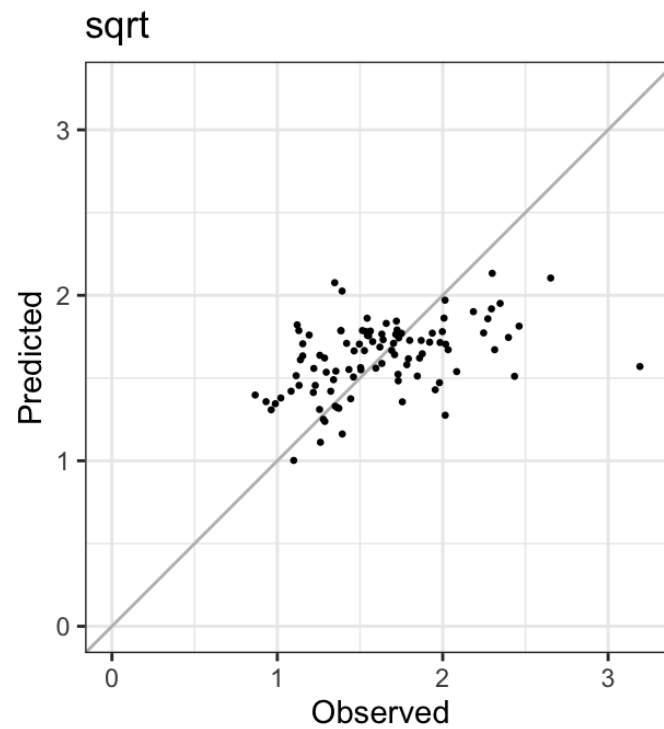
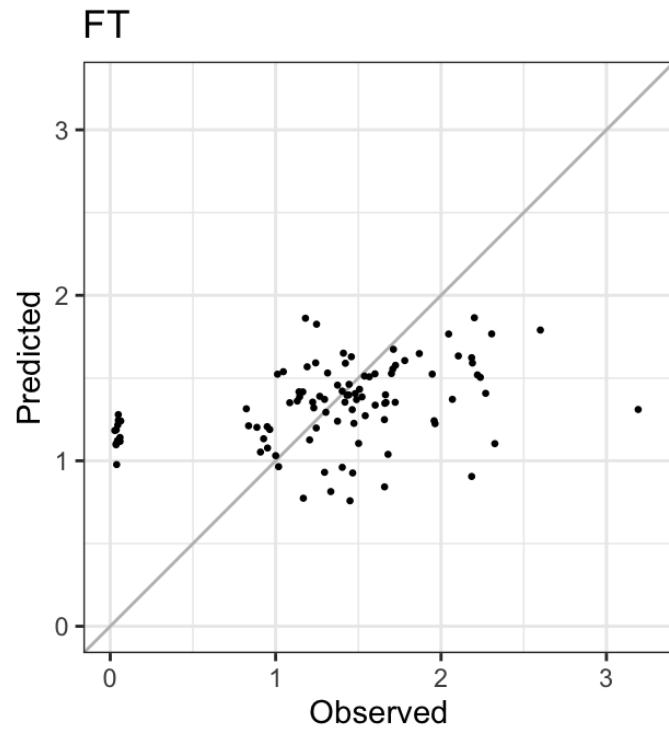
SAR (error)

```
1 nc_sar_err_ft    = spatialreg::spautolm(formula = FT ~ 1,    data = nc, listw = listW, family = "SAR")
2 nc_sar_err_sqrt  = spatialreg::spautolm(formula = sqrt ~ 1, data = nc, listw = listW, family = "SAR")
3 nc_sar_err_log   = spatialreg::spautolm(formula = log ~ 1,   data = nc, listw = listW, family = "SAR")
```

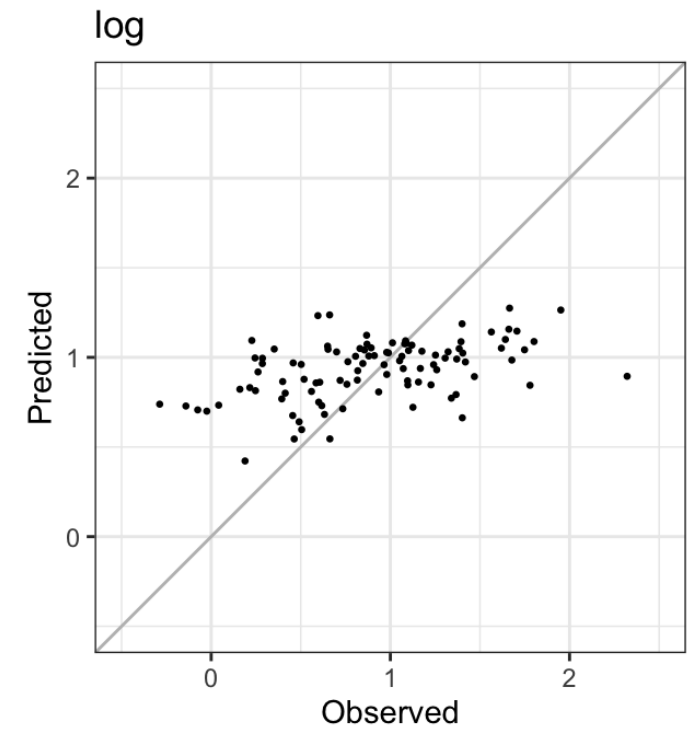
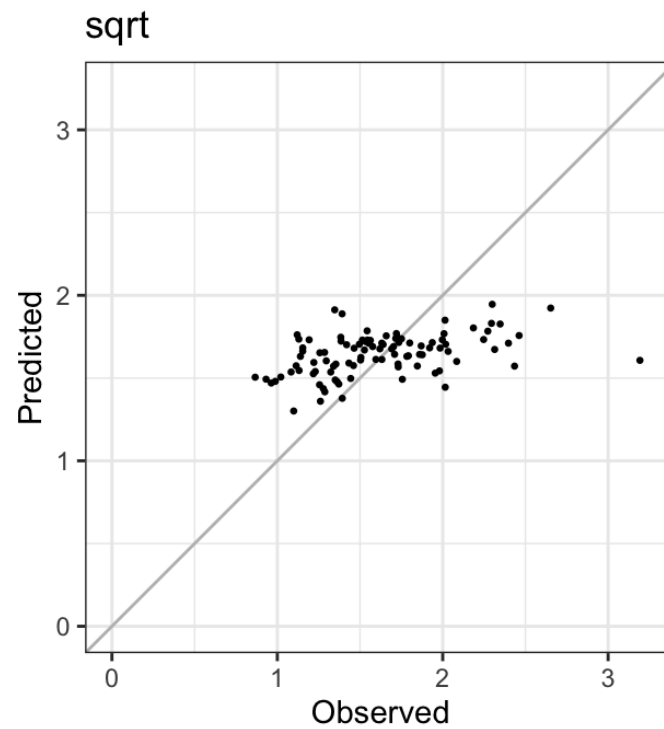
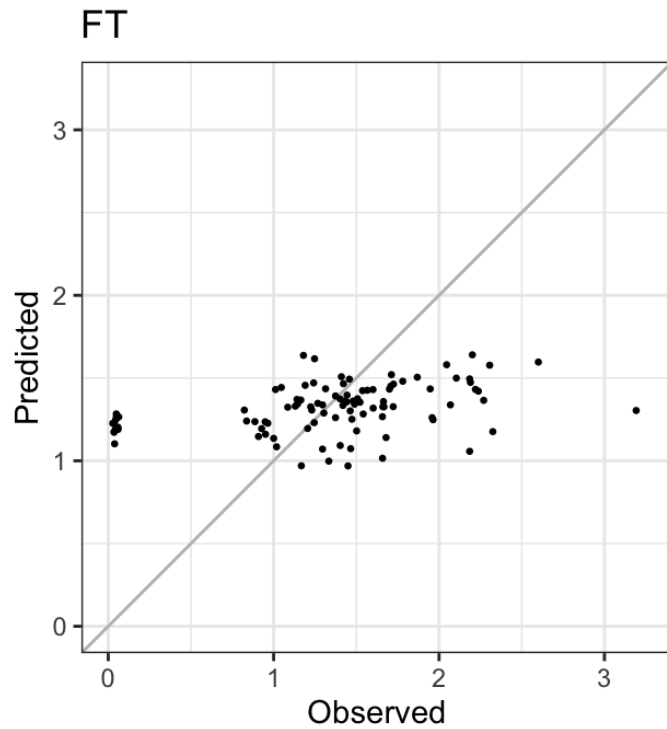
SAR (lag)

```
1 nc_sar_lag_ft    = spatialreg::lagsarlm(formula = FT ~ 1,    data = nc, listw = listW)
2 nc_sar_lag_sqrt  = spatialreg::lagsarlm(formula = sqrt ~ 1, data = nc, listw = listW)
3 nc_sar_lag_log   = spatialreg::lagsarlm(formula = log ~ 1,   data = nc, listw = listW)
```

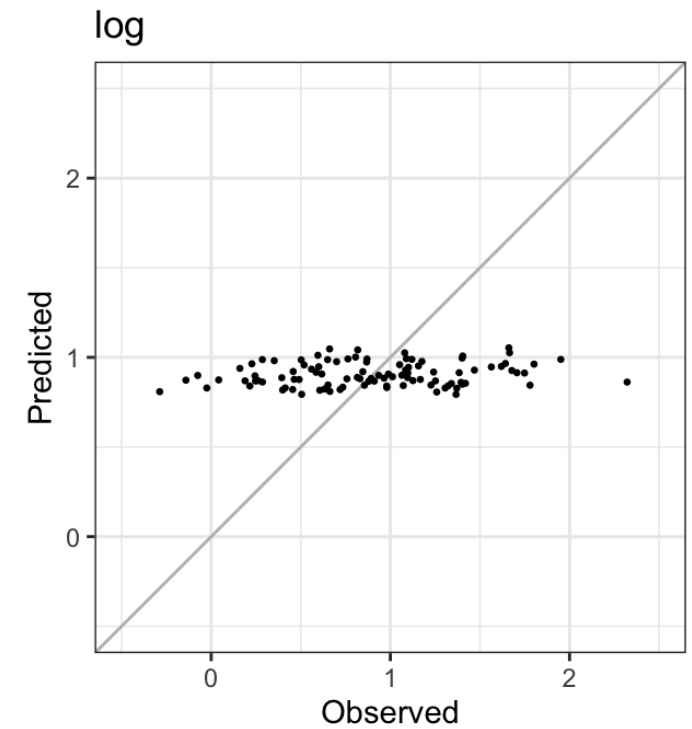
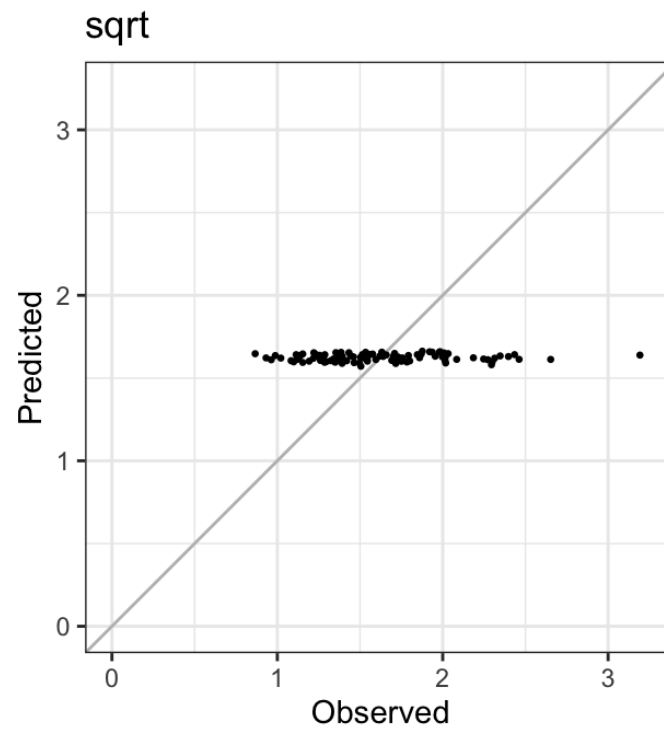
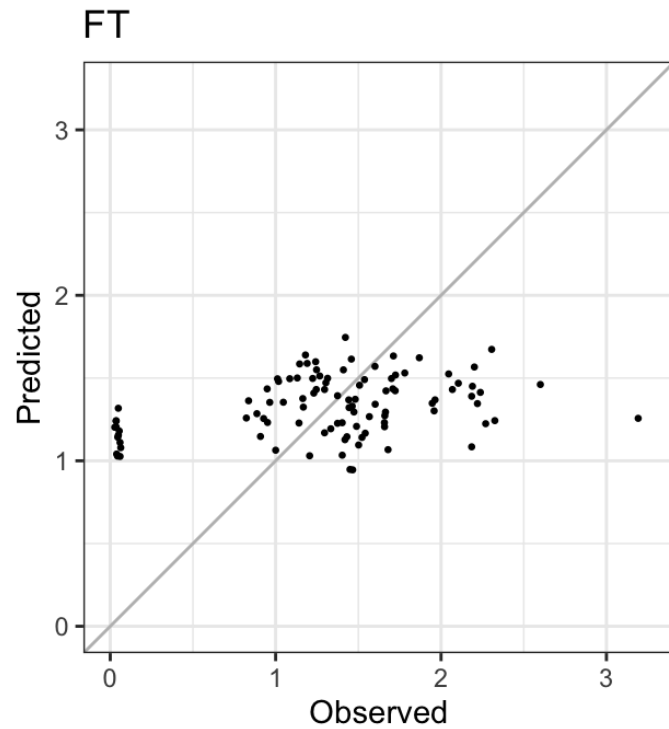
CAR predictions



SAR (error) predictions



SAR (lag) predictions



Residual spatial autocorrelation

```
1 spdep::moran.test(  
2   residuals(nc_car_sqrt), listW  
3 )
```

Moran I test under randomisation

```
data:  residuals(nc_car_sqrt)  
weights: listW
```

```
Moran I statistic standard deviate = -3.1196, p-  
value = 0.9991
```

```
alternative hypothesis: greater
```

```
sample estimates:
```

Moran I statistic	Expectation
-0.200890550	-0.010101010
0.003740354	

```
1 spdep::moran.test(  
2   residuals(nc_sar_err_sqrt), listW  
3 )
```

Moran I test under randomisation

```
data:  residuals(nc_sar_err_sqrt)  
weights: listW
```

```
Moran I statistic standard deviate = -0.422, p-  
value = 0.6635
```

```
alternative hypothesis: greater
```

```
sample estimates:
```

Moran I statistic	Expectation
-0.035976084	-0.010101010
0.003759585	

```
1 spdep::moran.test(  
2   residuals(nc_sar_lag_sqrt), listW  
3 )
```

Moran I test under randomisation

```
data:  residuals(nc_sar_lag_sqrt)  
weights: listW
```

```
Moran I statistic standard deviate = 4.7893, p-  
value = 8.368e-07
```

```
alternative hypothesis: greater
```

```
sample estimates:
```

```
Moran I statistic      Expectation
```

```
Variance
```

```
      0.285100348      -0.010101010  
0.003799187
```

CAR with brms (and Stan)

brms CAR

```
1 rownames(A) = nc$NAME
2 colnames(A) = nc$NAME
3
4 b_car = brms::brm(
5     1000*SID74/BIR74 ~ 1 + car(A, gr=NAME),
6     data=nc, data2=list(A=A),
7     control = list(adapt_delta = 0.95),
8     iter=20000,
9     cores = 4,
10    thin=10
11 )
```

```
1 b_car
```

Family: gaussian

Links: mu = identity; sigma = identity

Formula: 1000 * SID74/BIR74 ~ 1 + car(A, gr = NAME)

Data: nc (Number of observations: 100)

Draws: 4 chains, each with iter = 20000; warmup = 10000; thin = 10;
total post-warmup draws = 4000

Correlation Structures:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
car	0.74	0.16	0.37	0.97	1.00	1586	2099
sdcar	2.71	0.44	1.63	3.39	1.01	603	1685

Population-Level Effects:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	2.06	0.31	1.48	2.63	1.00	745	975

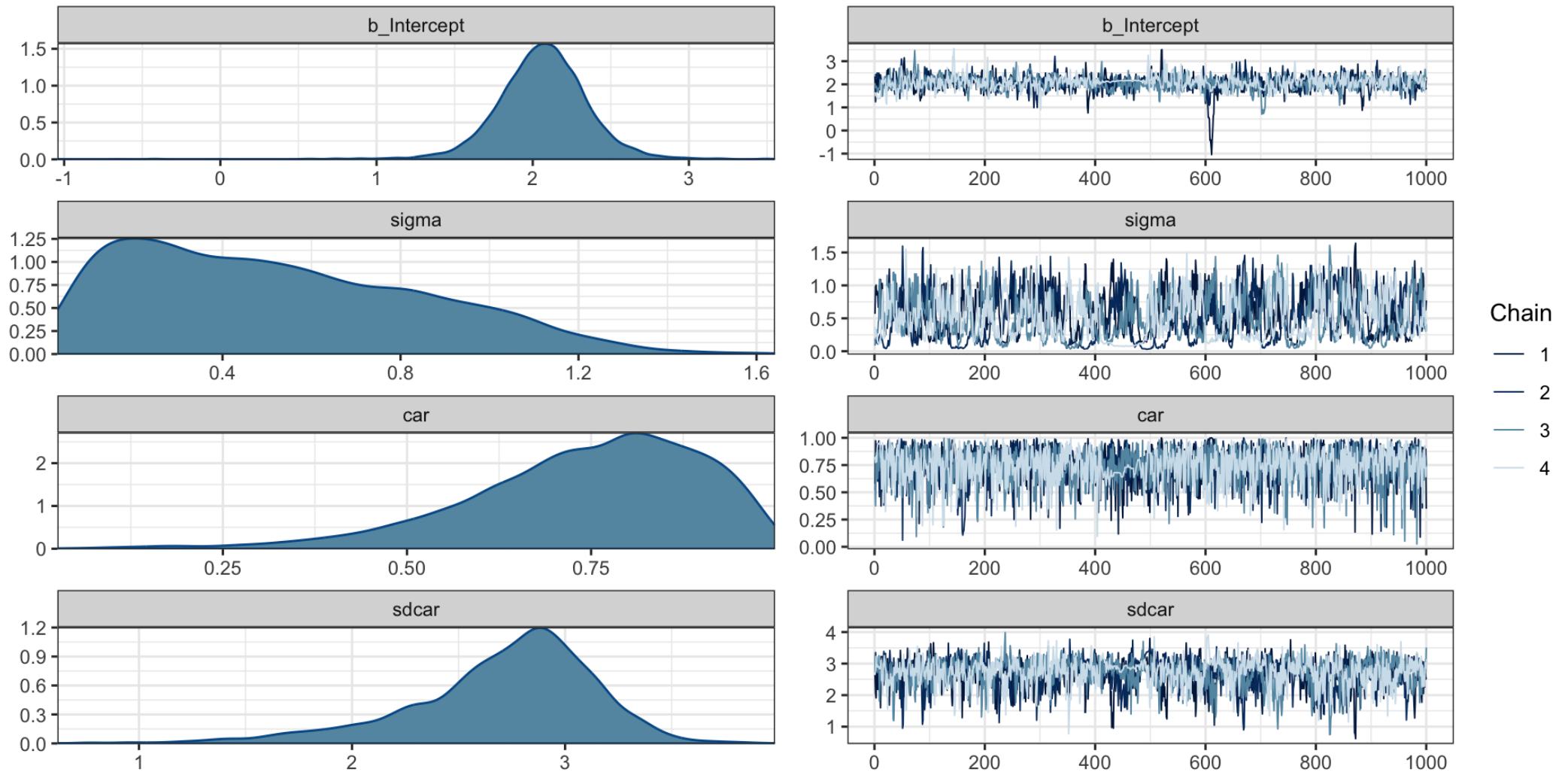
Family Specific Parameters:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sigma	0.52	0.33	0.06	1.21	1.02	218	201

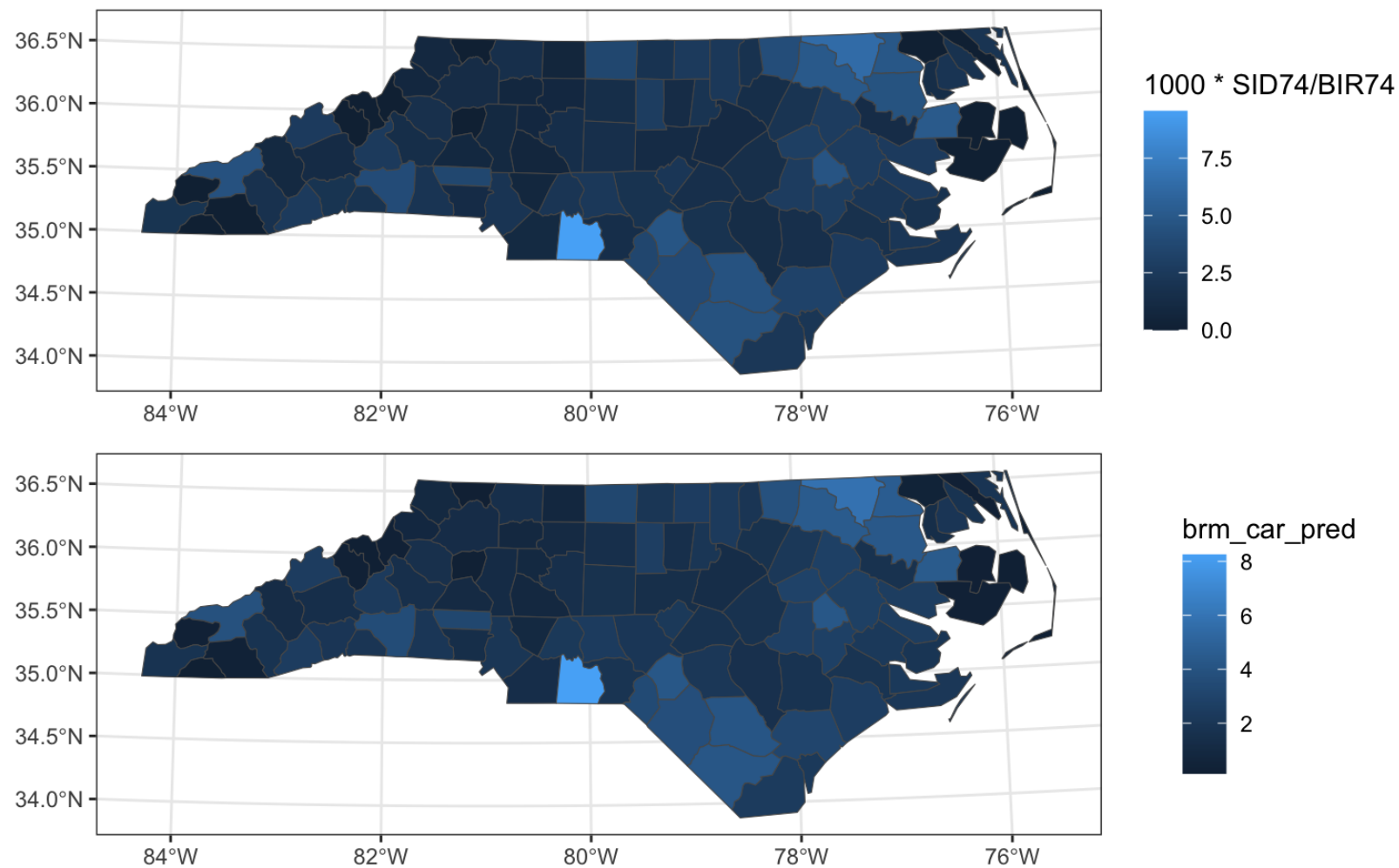
Draws were sampled using sampling(NUTS). For each parameter, Bulk_ESS and Tail_ESS are effective sample size measures, and Rhat is the potential

Diagnostics

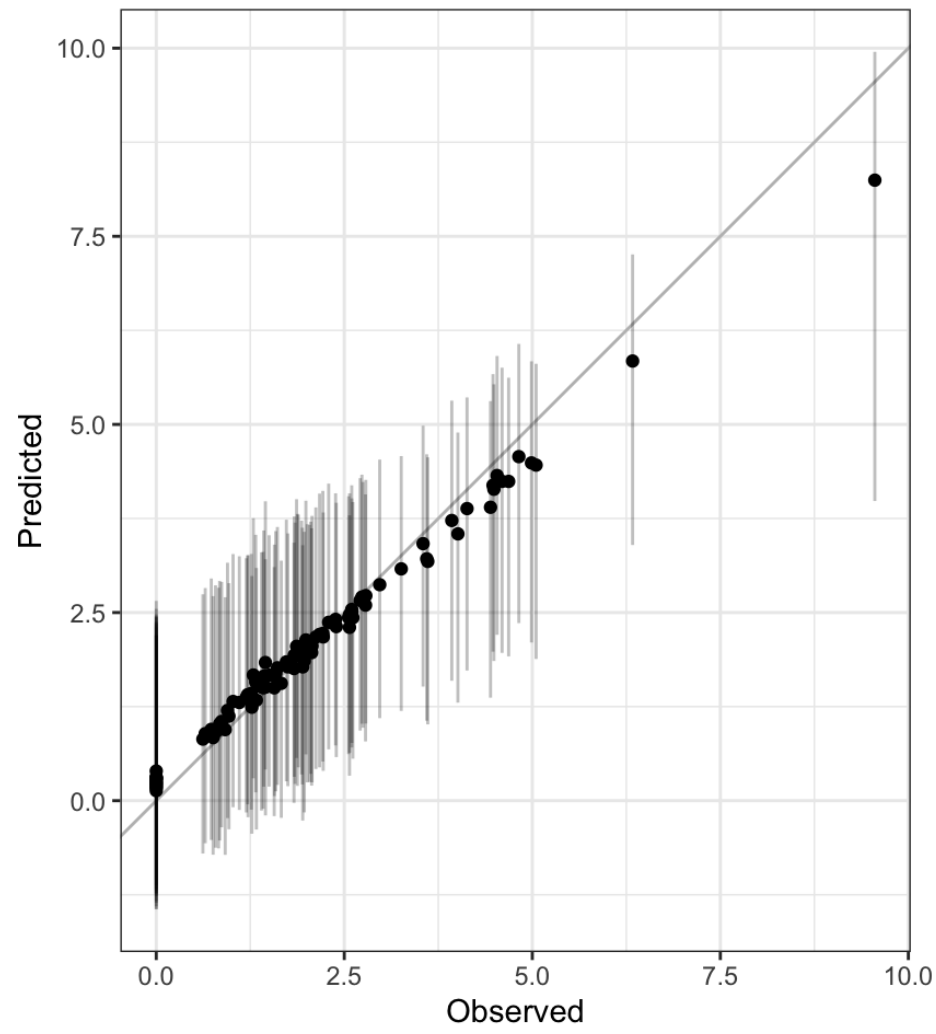
```
1 plot(b_car)
```



Predictions



Observed vs predicted



Stan (non-latent CAR)

```
1 stan_car = rstan::stan_model(  
2   model_code = "  
3     data {  
4       int<lower=1> n;  
5       int<lower=1> p;  
6       matrix[n, p] X;  
7       vector[n] y;  
8       matrix<lower=0, upper=1>[n, n] W;  
9       matrix[n, n] D;  
10      matrix[n, n] D_inv;  
11    }  
12    parameters {  
13      vector[p] beta;  
14      real<lower=0> tau;  
15      real<lower=0, upper=1> alpha;  
16    }  
17    transformed parameters {  
18      vector[n] y_cond = X * beta + alpha * I  
19      real<lower=0> sigma2 = 1/tau;  
20    }  
21    model {  
22      y ~ multi_normal_prec(X * beta + tau * I
```

```
1 X = model.matrix(~1, data = nc)  
2 d = list(  
3   n = nrow(X),           # number of obs  
4   p = ncol(X),           # number of coe  
5   X = X,                 # design matrix  
6   y = 1000*nc$SID74/nc$BIR74,  
7   W = A*1,  
8   D = diag(rowSums(A)),  
9   D_inv = diag(1/rowSums(A))  
10 )  
11  
12 s_car = rstan::sampling(stan_car, data = d, c
```

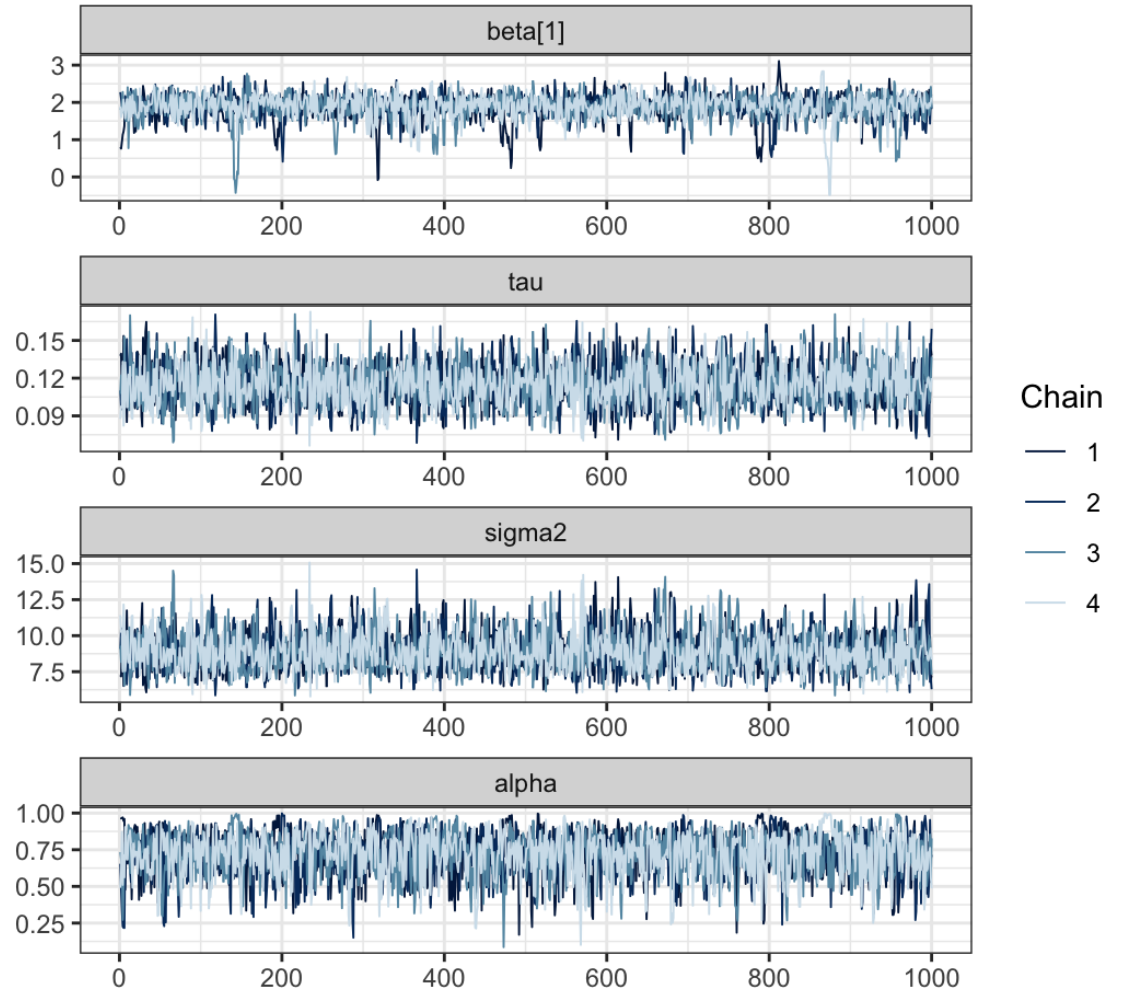
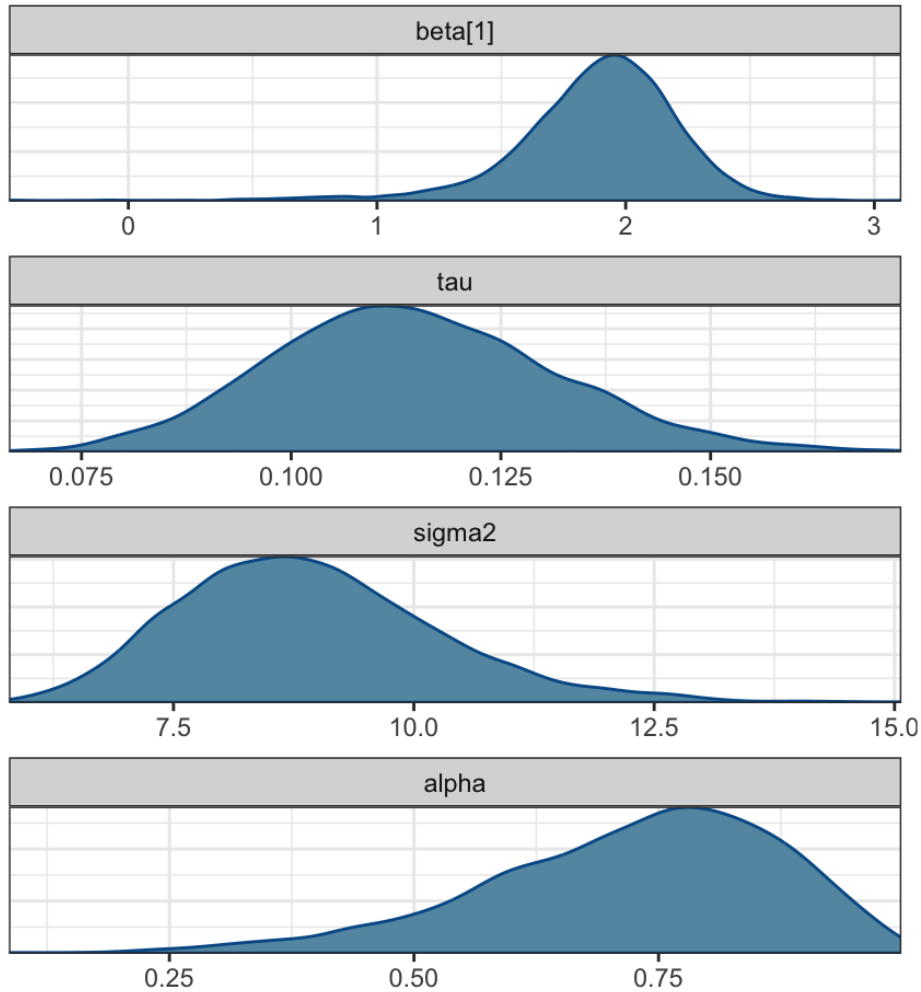
Results

```
1 rstan::extract(s_car, pars=c("beta[1]", "tau", "sigma2", "alpha")) |>
2   posterior::summarise_draws()
```

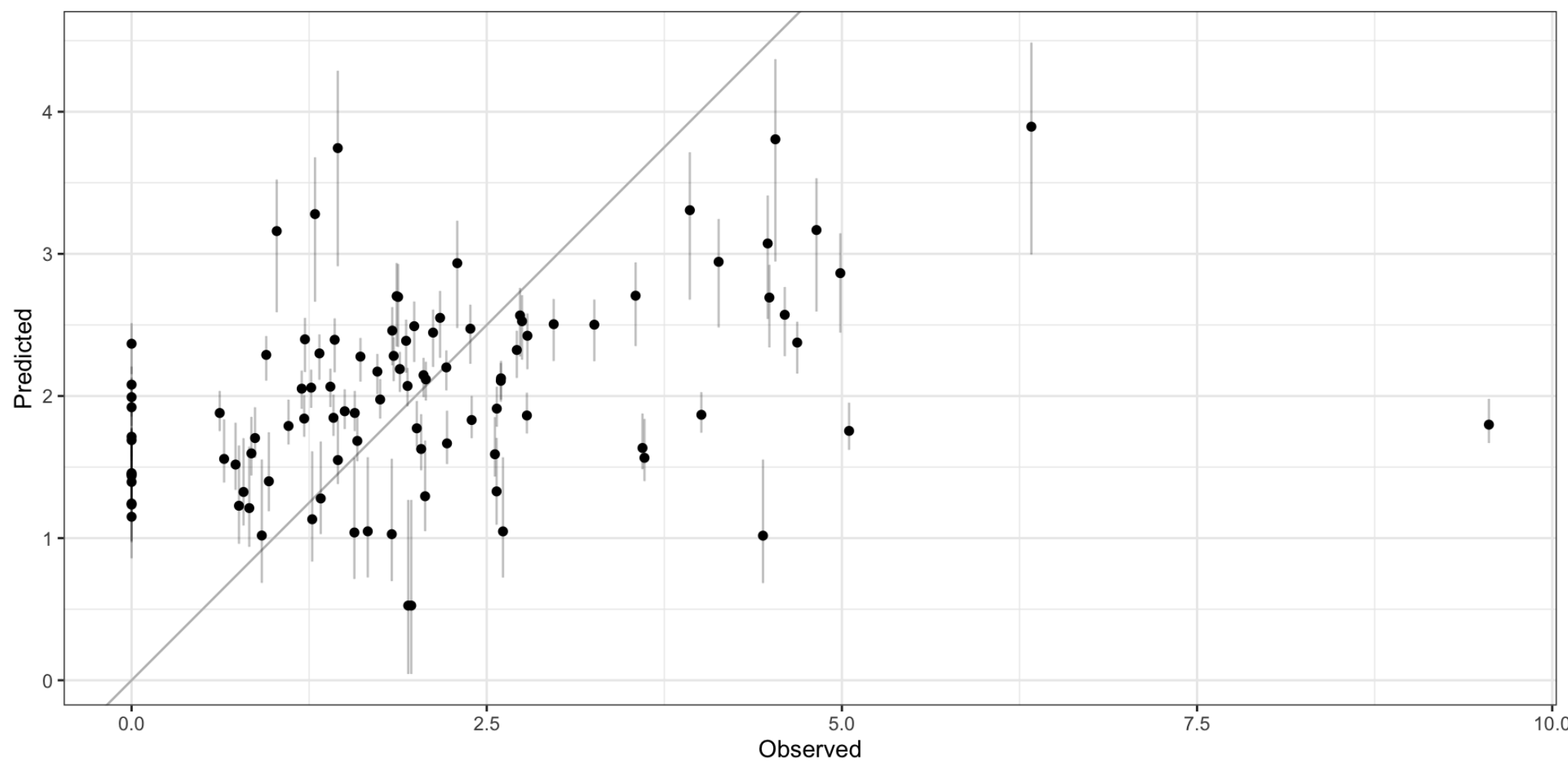
A tibble: 4 × 10

	variable	mean	median	sd	mad	q5	q95	rhat	ess_bulk	ess_tail
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	beta[1]	1.88	1.92	0.338	0.273	1.32	2.34	1.00	3764.	3983.
2	tau	0.115	0.114	0.0168	0.0168	0.0884	0.143	1.00	3957.	3830.
3	sigma2	8.92	8.80	1.34	1.29	6.98	11.3	1.00	3957.	3830.
4	alpha	0.723	0.744	0.150	0.150	0.439	0.932	1.00	4214.	3751.

Diagnostics



Observed vs predicted



What's different?

brms model

$$y \sim N(X\beta + w, \sigma^2)$$
$$w \sim N(0, \sigma_{CAR}^2 (D - \alpha W)^{-1})$$

Stan model

$$y \sim N(X\beta + w, \sigma^2 (D - \alpha W)^{-1})$$

SAR with brms

brms SAR (error)

```
1 W = diag(1/rowSums(A)) %*% A
2
3 b_sar_err = brms::brm(
4     1000*SID74/BIR74 ~ 1 + sar(W, type="error"),
5     data=nc, data2=list(W=W),
6     #silent=2, refresh=0,
7     iter=4000,
8     cores = 4,
9     thin = 2
10 )
```

```
1 b_sar_err
```

Family: gaussian

Links: mu = identity; sigma = identity

Formula: 1000 * SID74/BIR74 ~ 1 + sar(W, type = "error")

Data: nc (Number of observations: 100)

Draws: 4 chains, each with iter = 4000; warmup = 2000; thin = 2;
total post-warmup draws = 4000

Correlation Structures:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
errorsar	0.42	0.12	0.18	0.64	1.00	3379	3664

Population-Level Effects:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	2.04	0.27	1.49	2.56	1.00	3237	3239

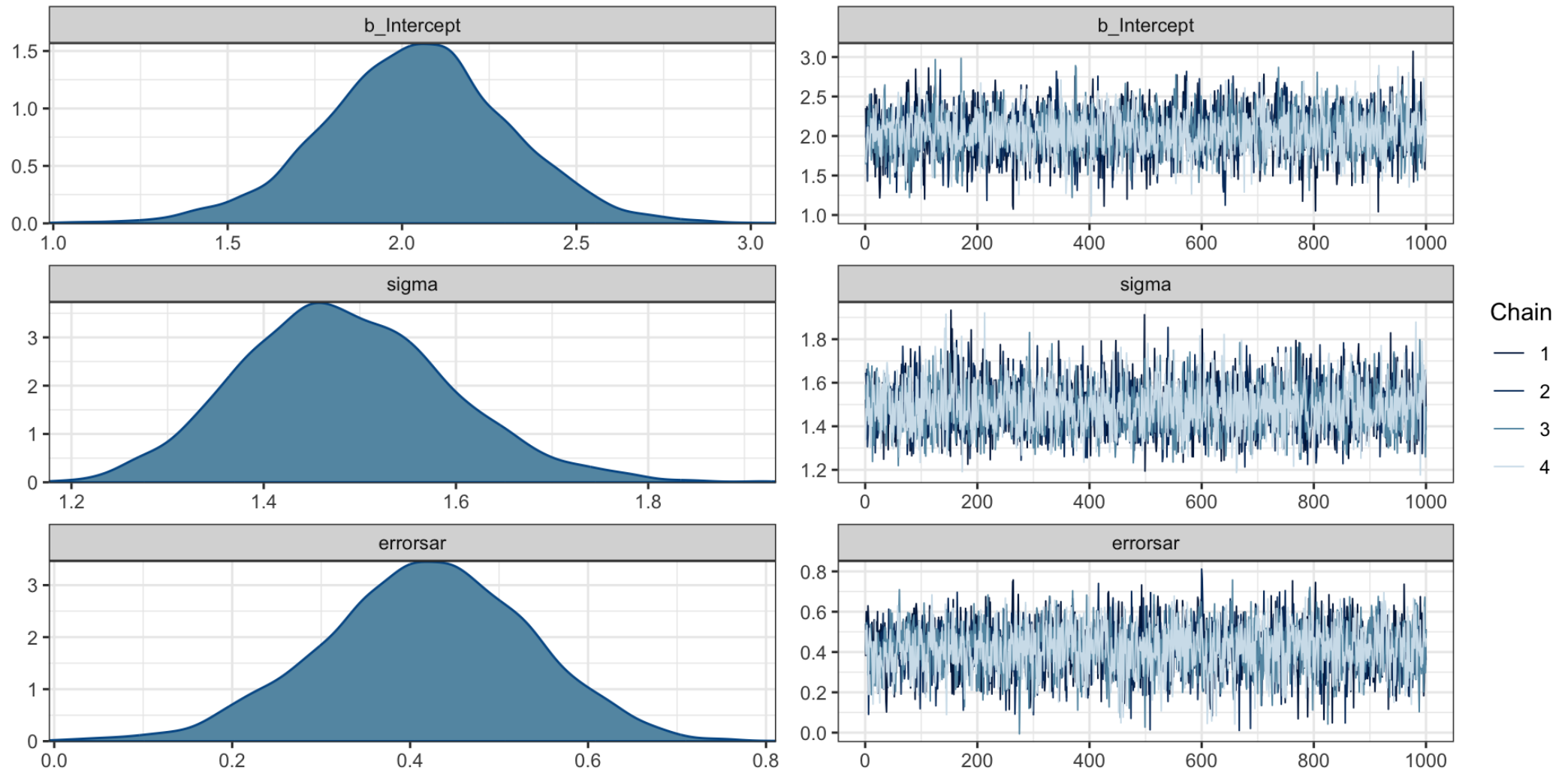
Family Specific Parameters:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sigma	1.49	0.11	1.29	1.72	1.00	3613	3691

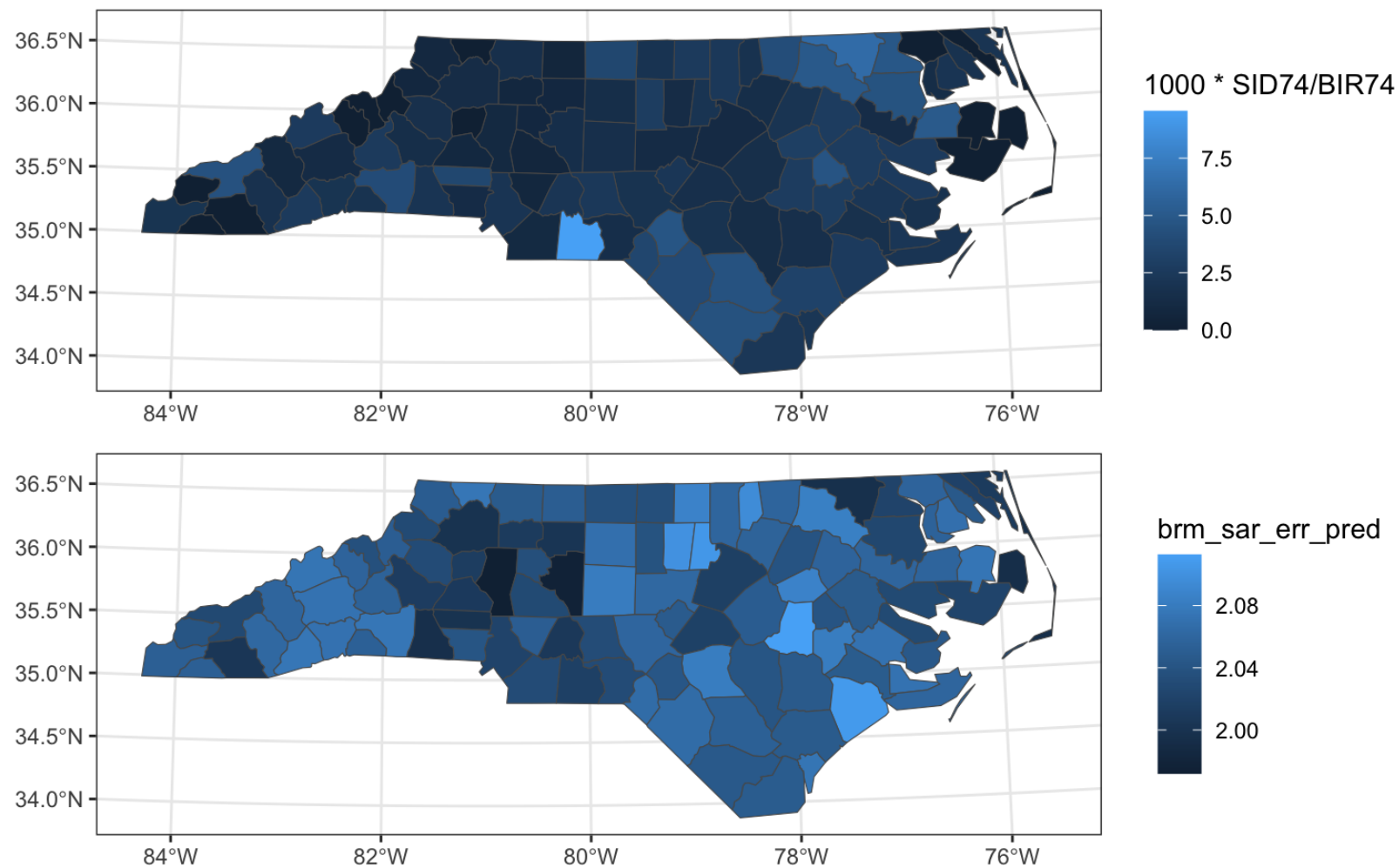
Draws were sampled using sampling(NUTS). For each parameter, Bulk_ESS and Tail_ESS are effective sample size measures, and Rhat is the potential scale reduction factor on split chains (at convergence, Rhat = 1).

Diagnostics

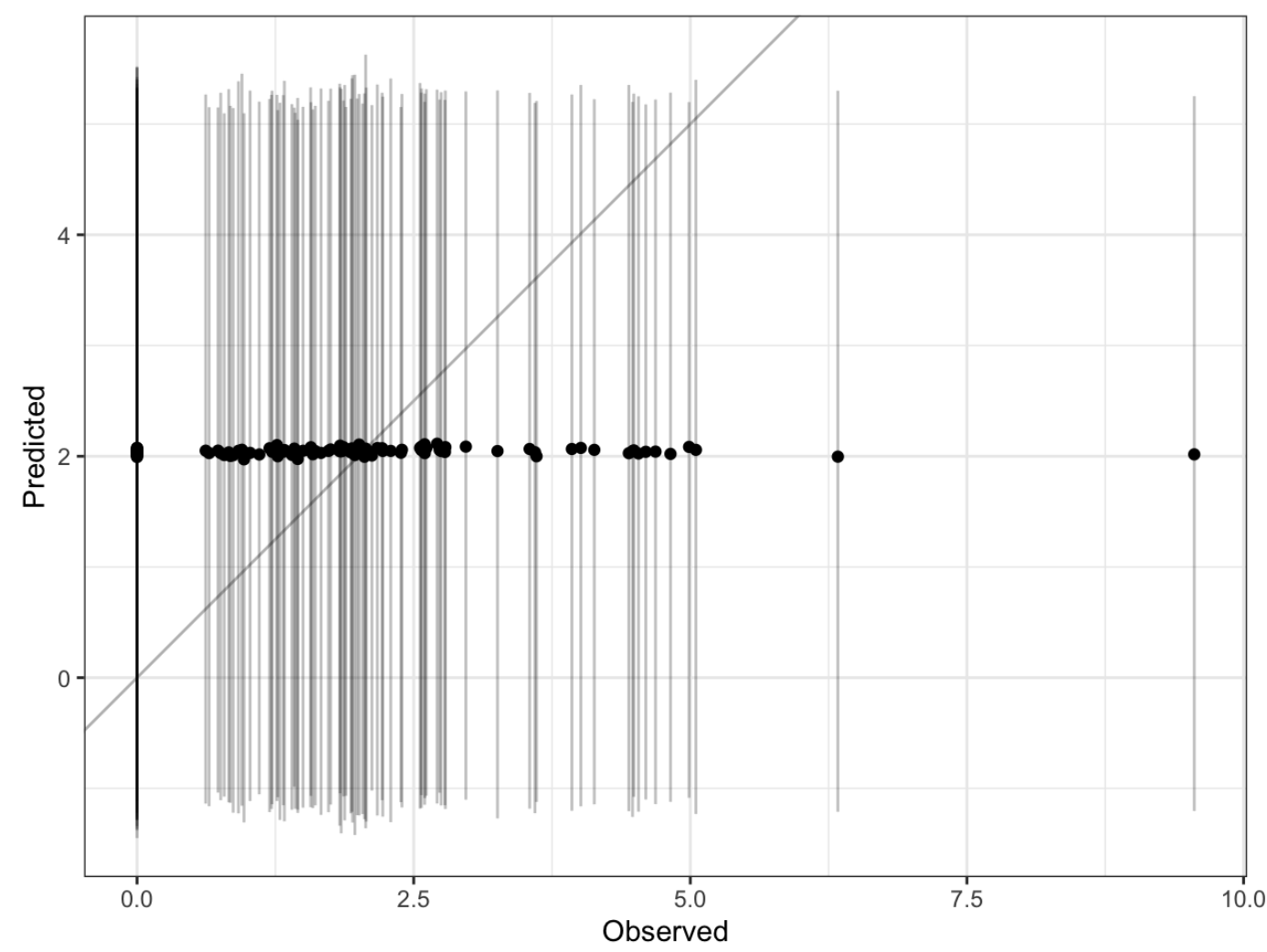
```
1 plot(b_sar_err)
```



Predictions

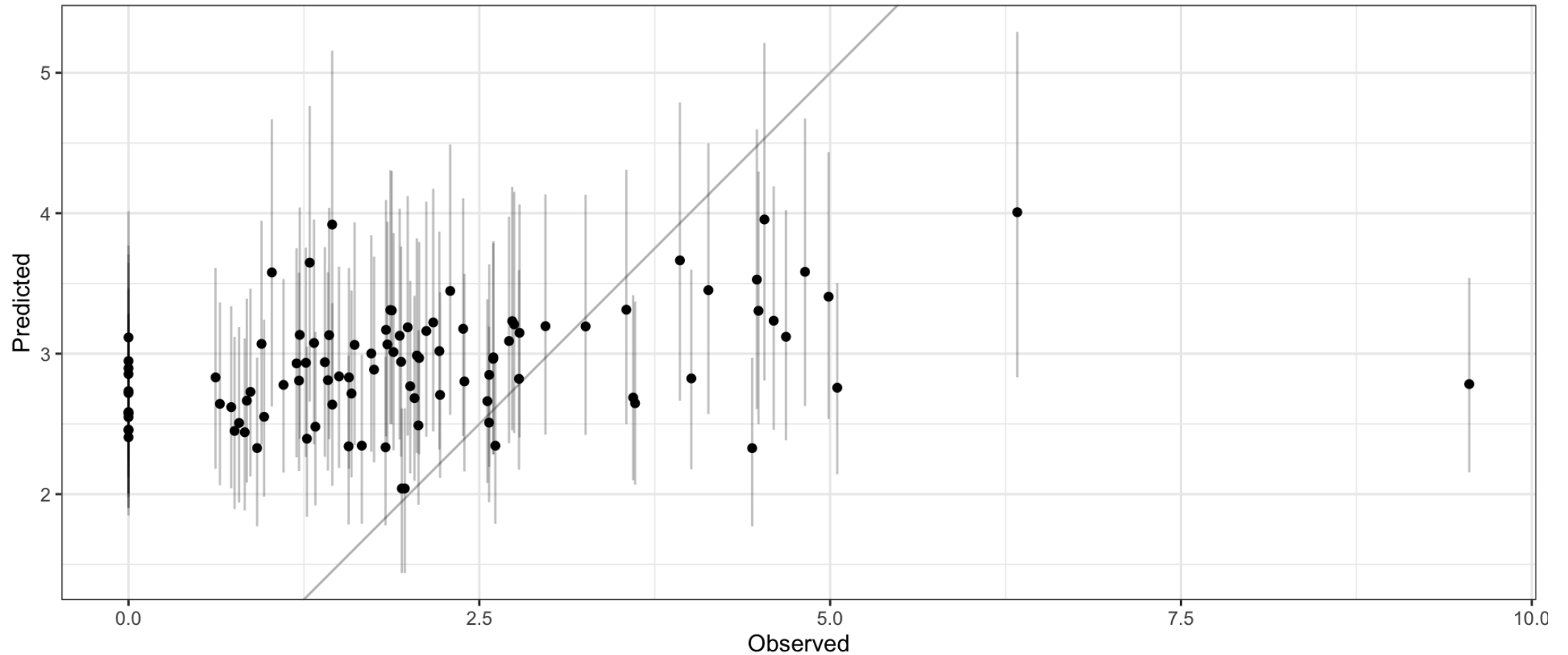


Observed vs predicted



Correcting `predict()`

If instead we use $\mathbf{X}\boldsymbol{\beta} + \phi\mathbf{W}\mathbf{y}$, we get the following:



brms SAR (lag)

```
1 W = diag(1/rowSums(A)) %*% A
2
3 b_sar_lag = brms::brm(
4     1000*SID74/BIR74 ~ 1 + sar(W, type="lag"),
5     data=nc, data2=list(W=W),
6     iter=4000,
7     cores = 4,
8     thin = 2
9 )
```

```
1 b_sar_lag
```

Family: gaussian

Links: mu = identity; sigma = identity

Formula: 1000 * SID74/BIR74 ~ 1 + sar(W, type = "lag")

Data: nc (Number of observations: 100)

Draws: 4 chains, each with iter = 4000; warmup = 2000; thin = 2;
total post-warmup draws = 4000

Correlation Structures:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
lagsar	0.38	0.12	0.14	0.60	1.00	2769	2940

Population-Level Effects:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
Intercept	1.28	0.29	0.74	1.85	1.00	2860	2779

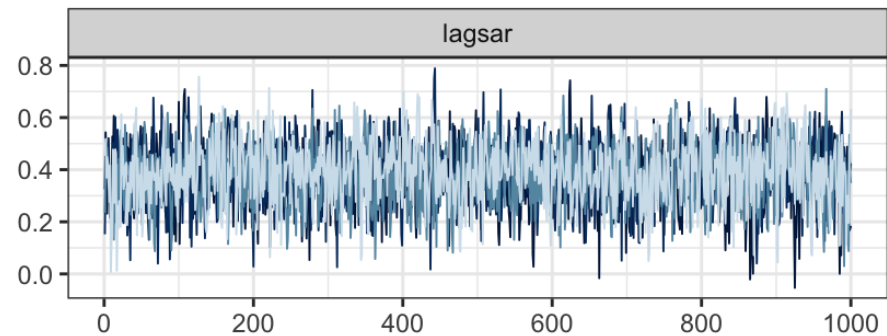
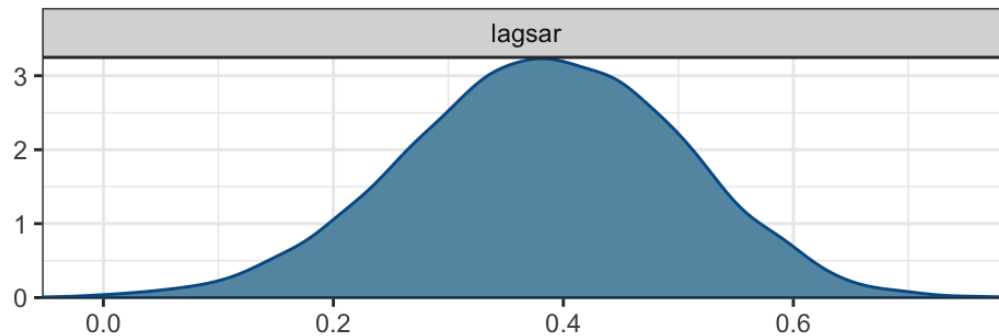
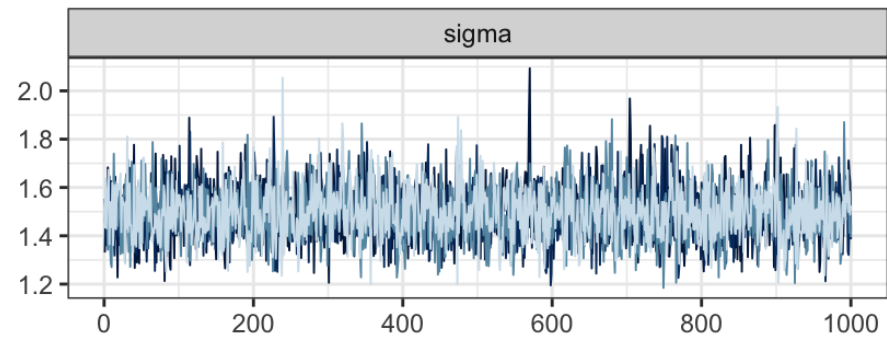
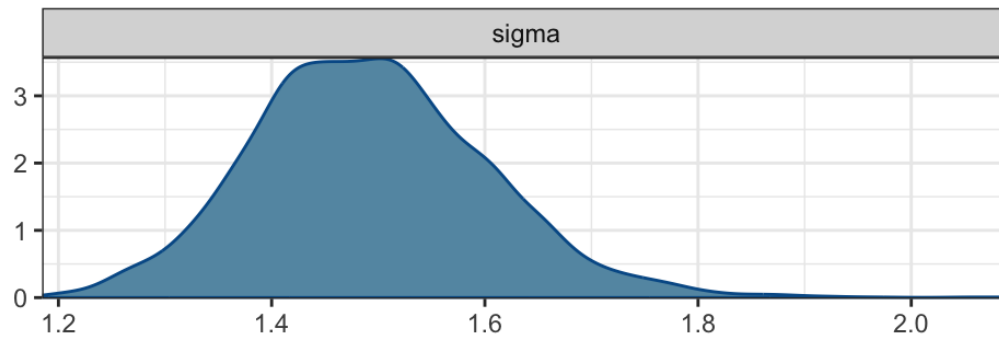
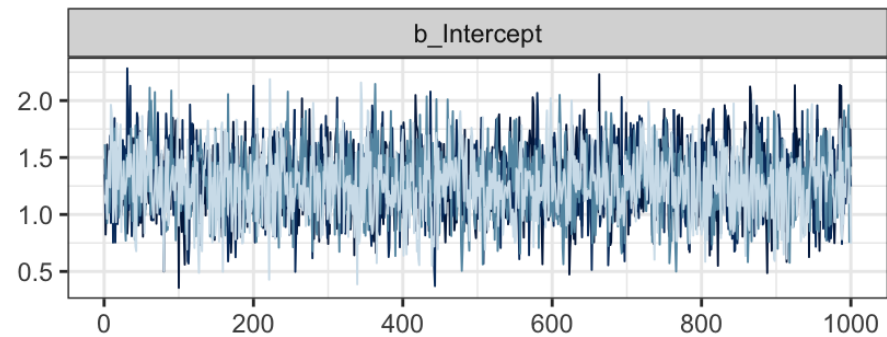
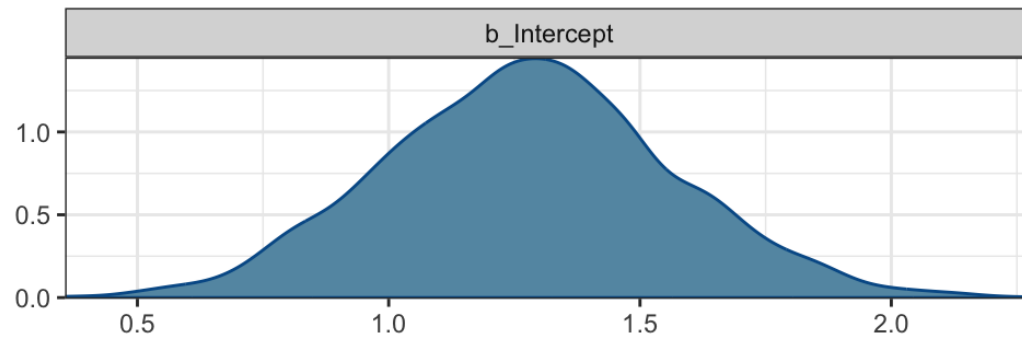
Family Specific Parameters:

	Estimate	Est.Error	l-95% CI	u-95% CI	Rhat	Bulk_ESS	Tail_ESS
sigma	1.49	0.11	1.29	1.73	1.00	3336	3298

Draws were sampled using sampling(NUTS). For each parameter, Bulk_ESS and Tail_ESS are effective sample size measures, and Rhat is the potential scale reduction factor on split chains (at convergence, Rhat = 1).

Diagnostics

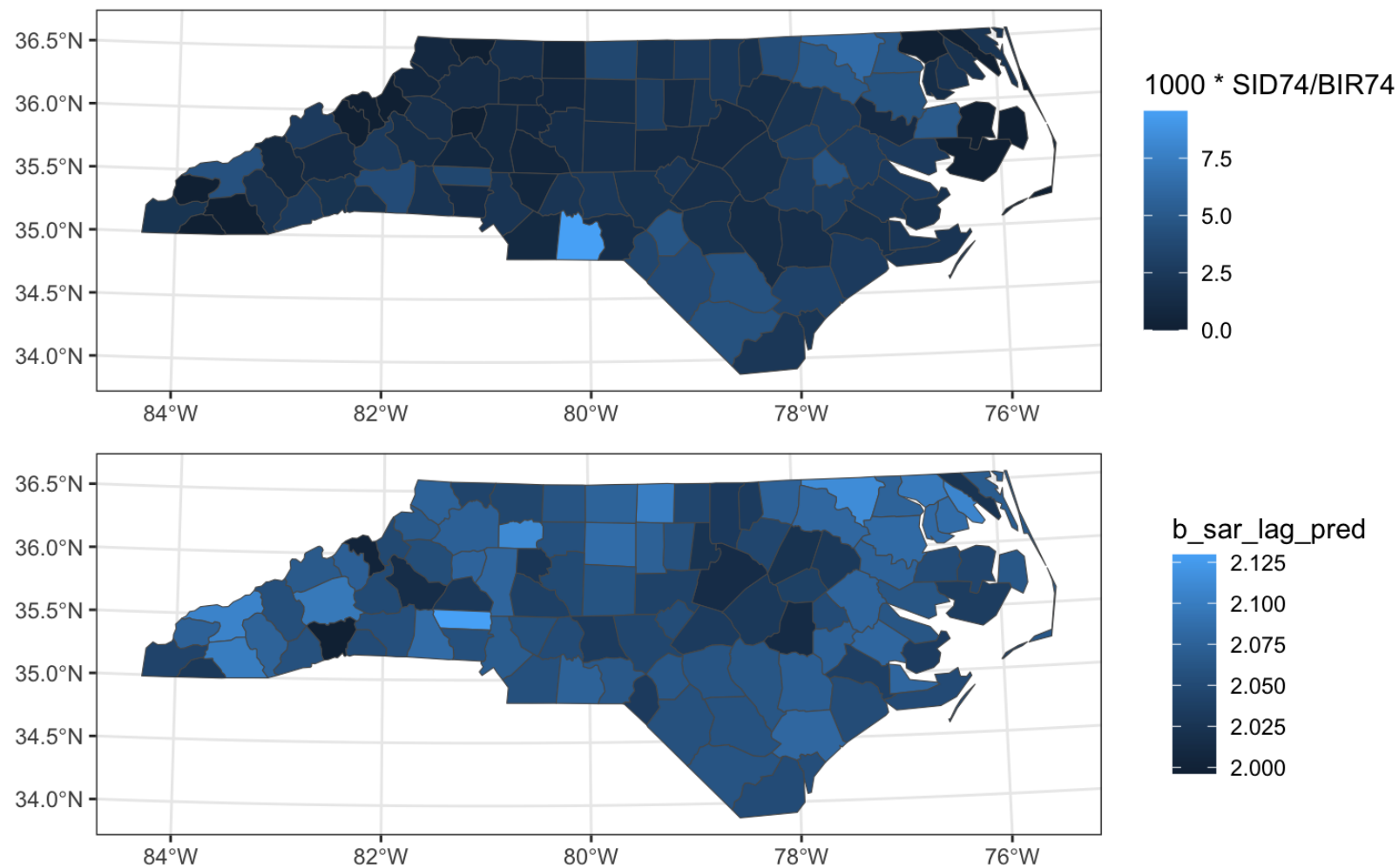
```
1 plot(b_sar_lag)
```



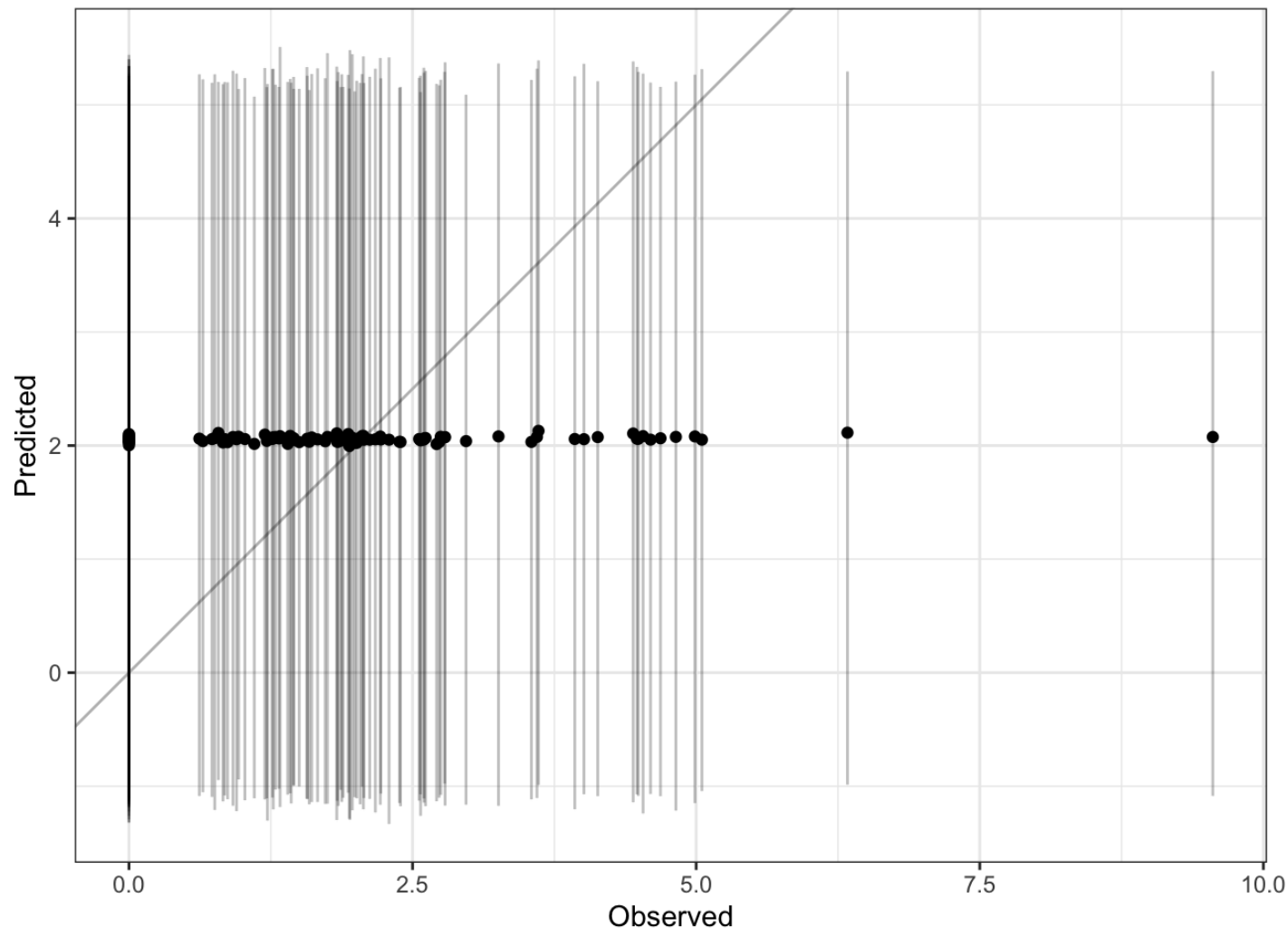
Chain

— 1
— 2
— 3
— 4

Predictions



Observed vs predicted



Correcting `predict()`

If instead we use $\mathbf{X}\boldsymbol{\beta} + \phi\mathbf{W}\mathbf{y}$, we get the following:

