

Seasonal Arima

Lecture 11

Dr. Colin Rundel

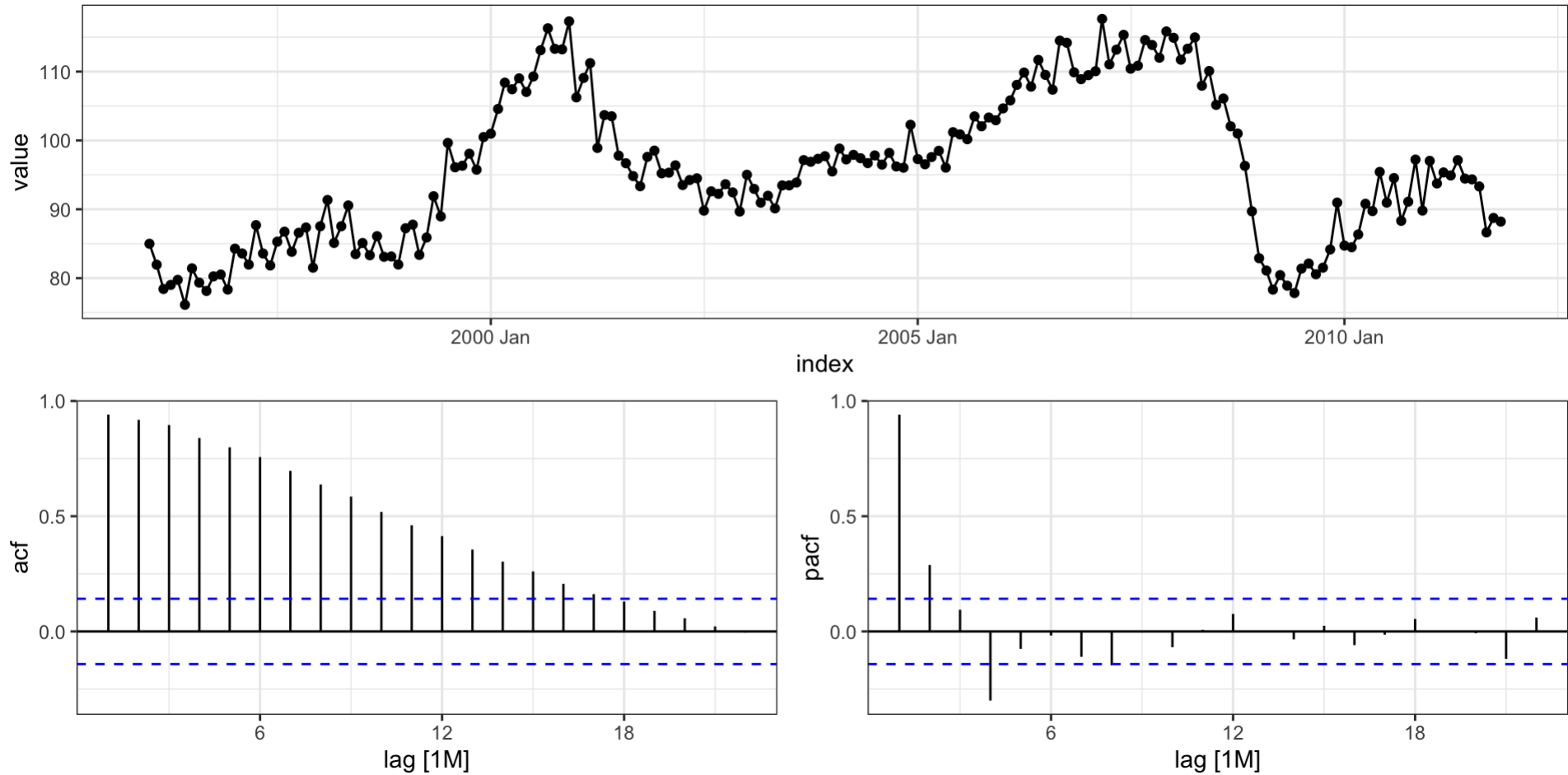
ARIMA - General Guidance

1. Positive autocorrelations out to a large number of lags usually indicates a need for differencing
2. Slightly too much or slightly too little differencing can be corrected by adding AR or MA terms respectively.
3. A model with no differencing usually includes a constant term, a model with two or more orders (rare) differencing usually does not include a constant term.
4. After differencing, if the PACF has a sharp cutoff then consider adding AR terms to the model.
5. After differencing, if the ACF has a sharp cutoff then consider adding an MA term to the model.
6. It is possible for an AR term and an MA term to cancel each other's effects, so try models with fewer AR terms and fewer MA terms.

Electrical Equipment Sales

Data

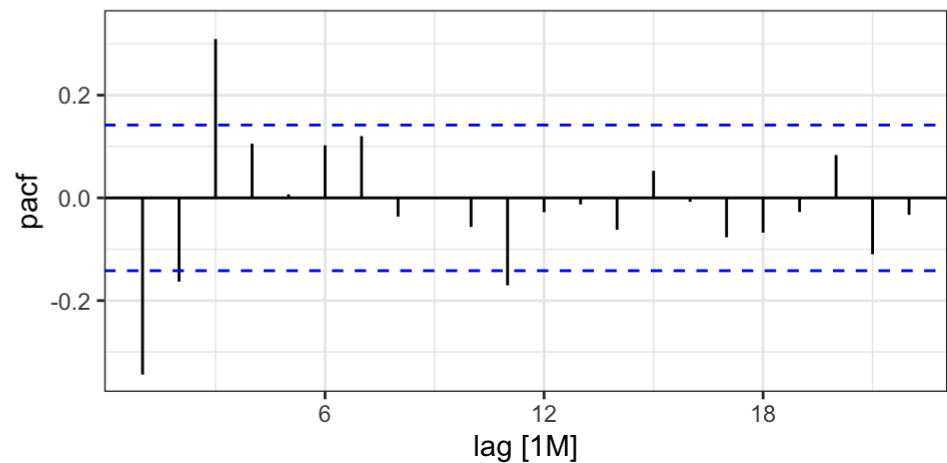
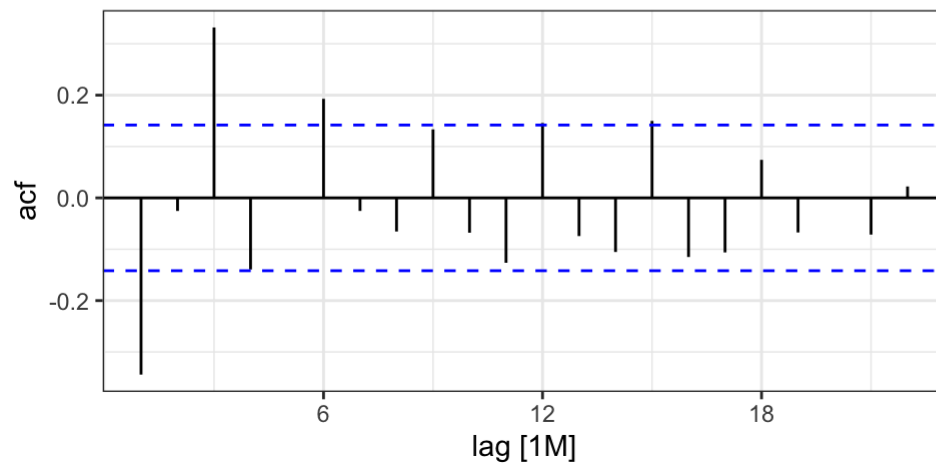
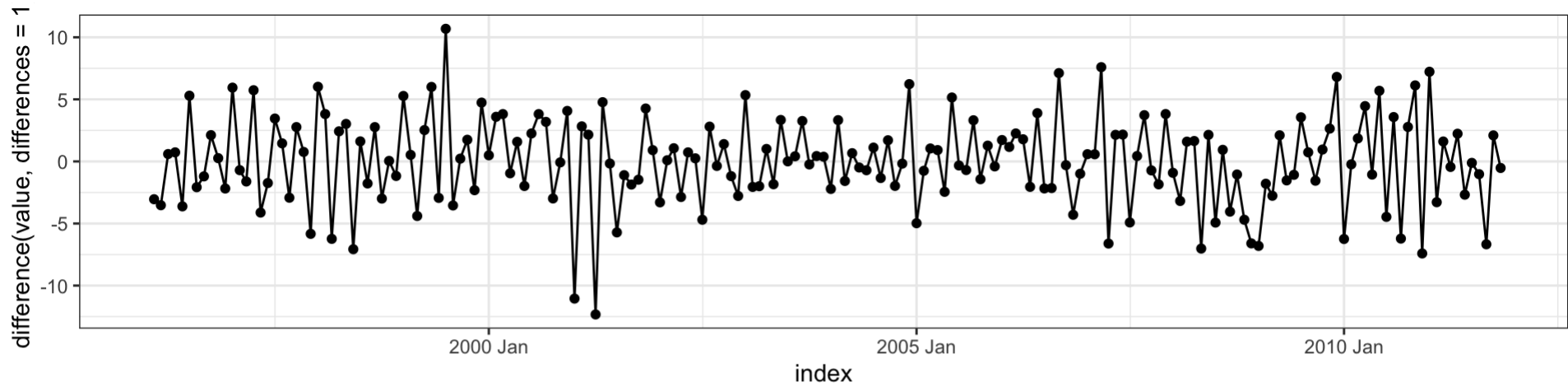
```
1 feasts::gg_tsdisplay(elec_sales, y=value, plot_type="partial")
```



Differencing

differences = 1 differences = 2

```
1 feasts::gg_tsdisplay(elec_sales, y=difference(value, differences = 1), plot_type="partial")
```



Model

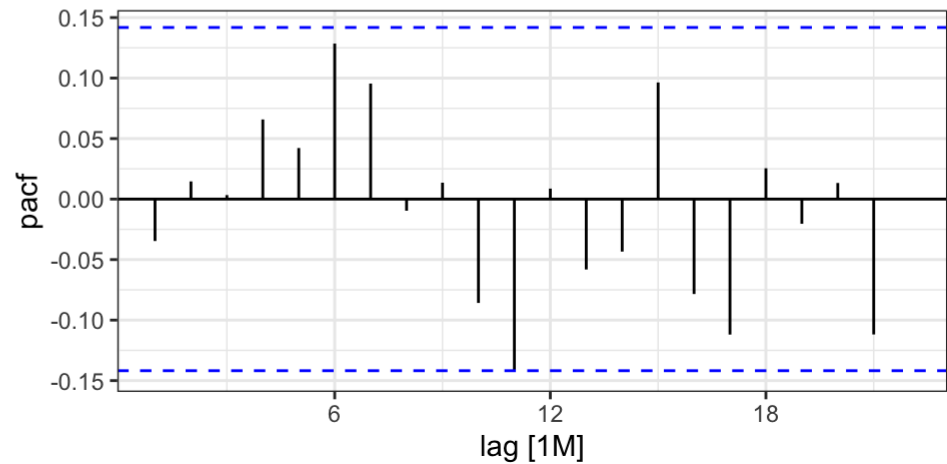
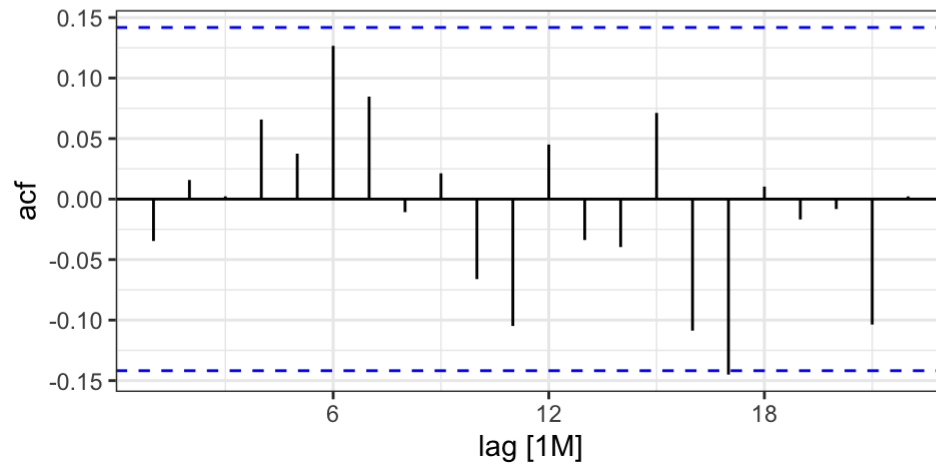
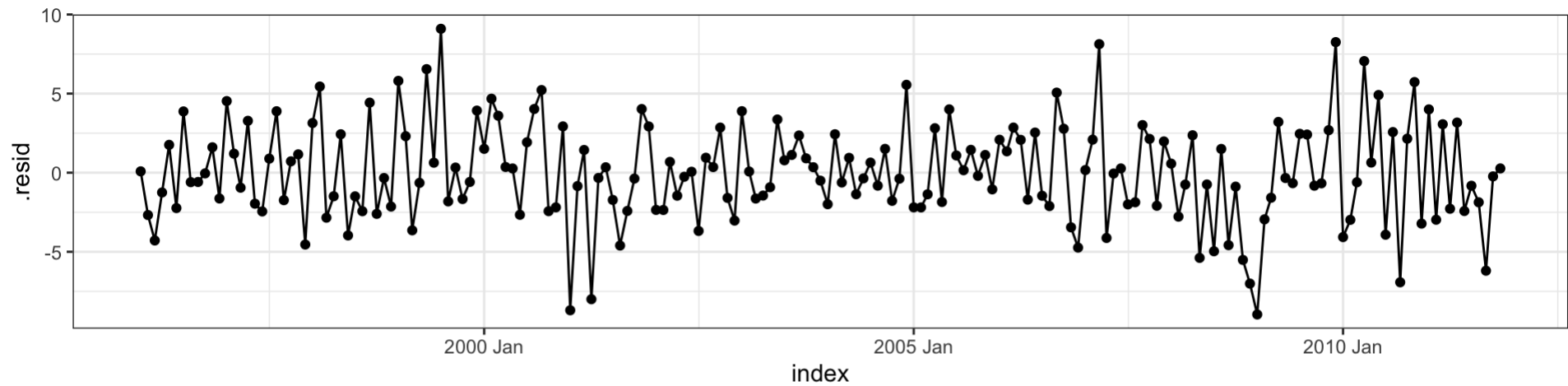
```
1 m = model(  
2   elec_sales,  
3   ARIMA(value ~ pdq(3,1,0))  
4 )  
5  
6 glance(m)
```

```
# A tibble: 1 × 8
```

.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots
ma_roots						
<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>
<list>						
1 ARIMA(value ~ pdq(3, 1, 0))	9.85	-486.	979.	980.	992.	<cpl>
<cpl>						

Residuals

```
1 residuals(m) |>  
2 feasts::gg_tsdisplay(y=.resid, plot_type="partial")
```



Information Criteria

The fable package provides a number of different information criteria for model selection,

$$AIC = -2 \log \square + 2(p + q + k + 1)$$

$$AICc = AIC + \frac{2(p + q + k + 1)(p + q + k + 2)}{n - p - q - k - 2}$$

$$BIC = AIC + (\log(n) - 2)(p + q + k + 1)$$

For small values of n , AIC can overfit the data (select too many predictors) and so AICc is preferred. BIC is a more conservative version of AIC that penalizes the number of predictors more heavily.

Note that since differencing changes the data we are fitting the ARMA model to, it does not make sense to

Model Comparison

Model choices:

```
1 model(  
2   elec_sales,  
3   ARIMA(value ~ pdq(3,1,0)),  
4   ARIMA(value ~ pdq(2,1,0)),  
5   ARIMA(value ~ pdq(4,1,0)),  
6   ARIMA(value ~ pdq(3,1,1))  
7 ) |>  
8   glance()
```

A tibble: 4 × 8

	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	ARIMA(value ~ pdq(3, 1, 0))	9.85	-486.	979.	980.	992.	<cpl>	<cpl>
2	ARIMA(value ~ pdq(2, 1, 0))	10.9	-495.	997.	997.	1006.	<cpl>	<cpl>
3	ARIMA(value ~ pdq(4, 1, 0))	9.78	-484.	979.	979.	995.	<cpl>	<cpl>
4	ARIMA(value ~ pdq(3, 1, 1))	9.74	-484.	978.	978.	994.	<cpl>	<cpl>

Automatic selection (AICc)

```
1 model(  
2   elec_sales,  
3   ARIMA(value)  
4 ) |>  
5   report()
```

Series: value

Model: ARIMA(3,1,1)

Coefficients:

	ar1	ar2	ar3	ma1
	0.0519	0.1191	0.3730	-0.4542
s.e.	0.1840	0.0888	0.0679	0.1993

sigma^2 estimated as 9.737: log likelihood=-484.08

AIC=978.17 AICc=978.49 BIC=994.4

Automatic selection (AIC)

```
1 model(  
2   elec_sales,  
3   ARIMA(value, ic = "aic")  
4 ) |>  
5   report()
```

Series: value

Model: ARIMA(3,1,1)

Coefficients:

	ar1	ar2	ar3	ma1
	0.0519	0.1191	0.3730	-0.4542
s.e.	0.1840	0.0888	0.0679	0.1993

sigma² estimated as 9.737: log likelihood=-484.08

AIC=978.17 AICc=978.49 BIC=994.4

Automatic selection (BIC)

```
1 model(  
2   elec_sales,  
3   ARIMA(value, ic = "bic")  
4 ) |>  
5   report()
```

Series: value

Model: ARIMA(1,1,2)

Coefficients:

	ar1	ma1	ma2
	0.7738	-1.2298	0.5106
s.e.	0.0933	0.1035	0.0695

sigma² estimated as 10.2: log likelihood=-488.99

AIC=985.97 AICc=986.19 BIC=998.96

Model fit

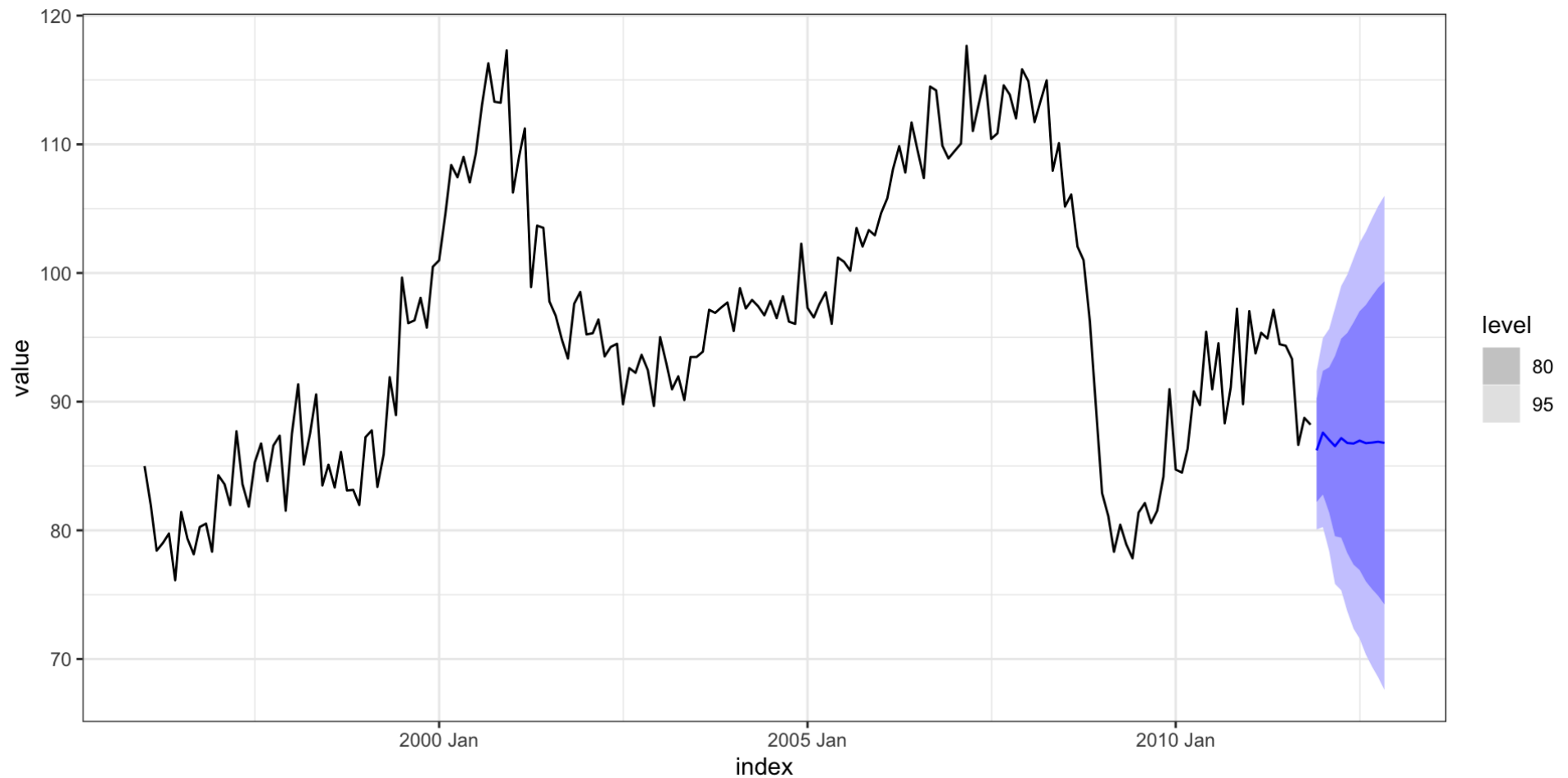
Fitted

Obs vs fitted

```
1 augment(m) |>
2 ggplot(aes(x=as.Date(index))) +
3   geom_line(aes(y=value), color = "black") +
4   geom_line(aes(y=.fitted), color = "red", alpha=0.75)
```

Model forecast

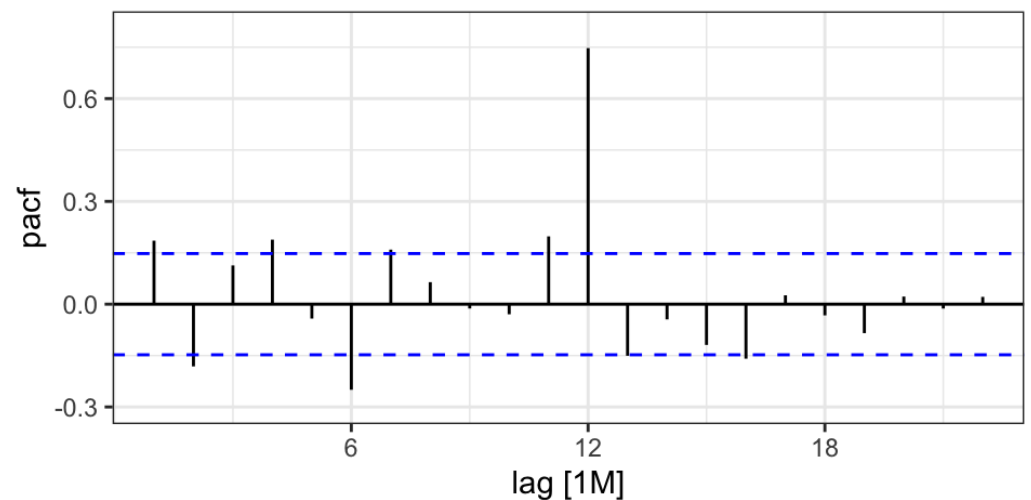
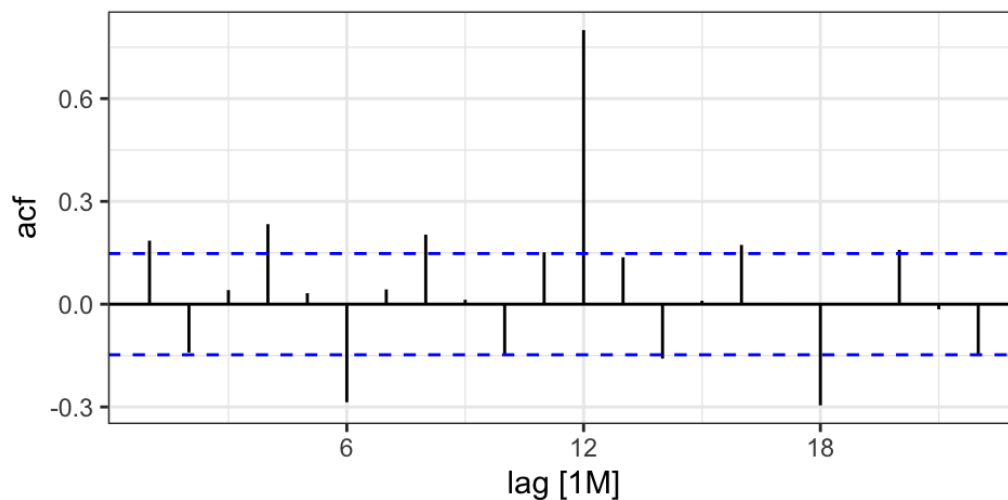
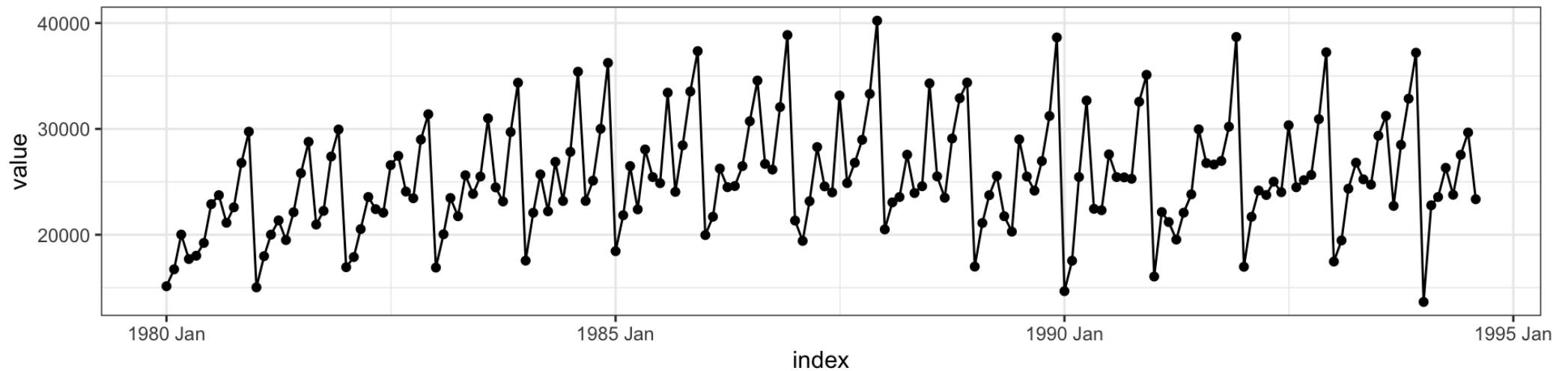
```
1 m |>  
2   forecast(h=12) |>  
3   autoplot(elec_sales)
```



Seasonal Models

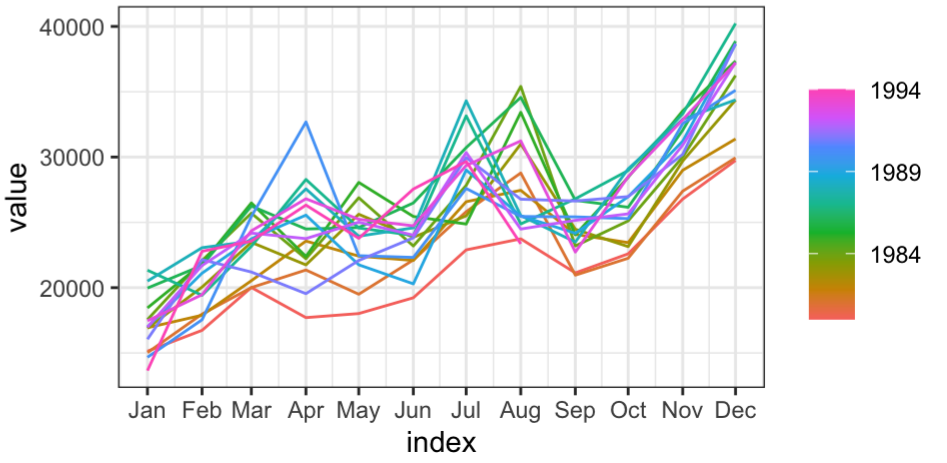
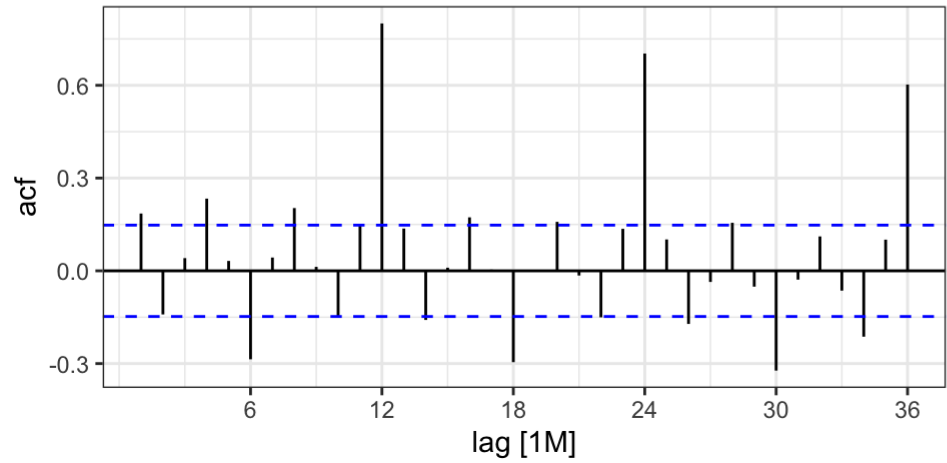
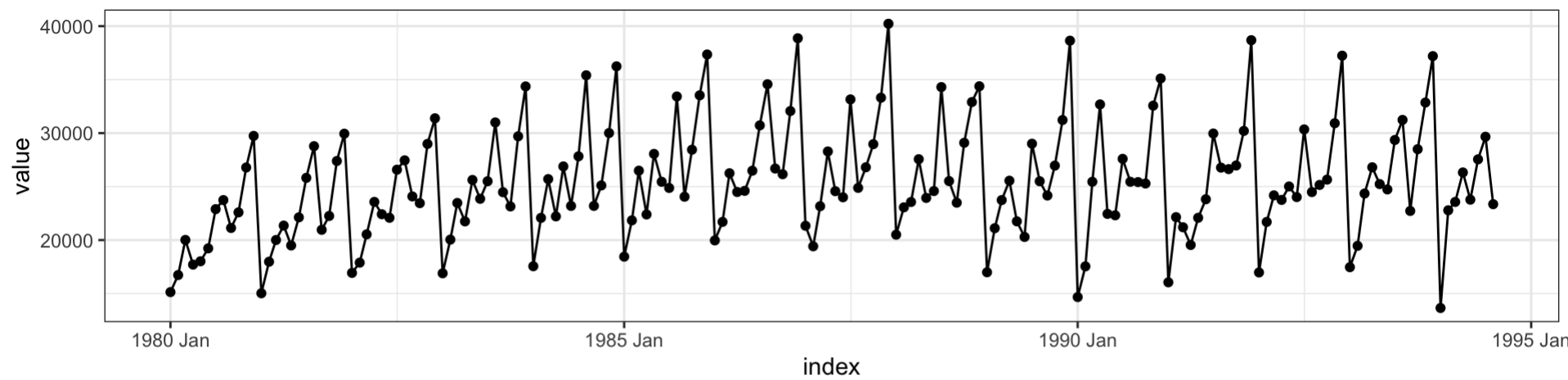
Australian Wine Sales Example

Australian total wine sales by wine makers in bottles ≤ 1 litre. Jan 1980 – Aug 1994.



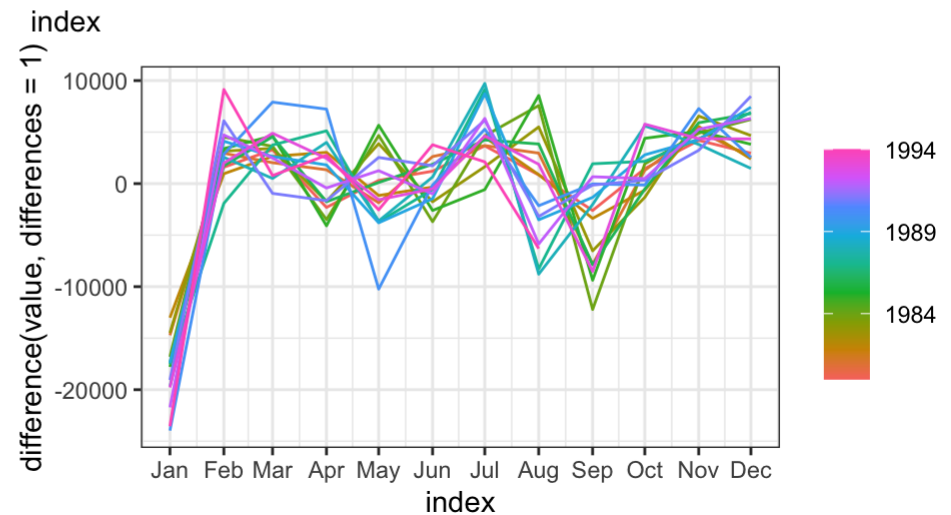
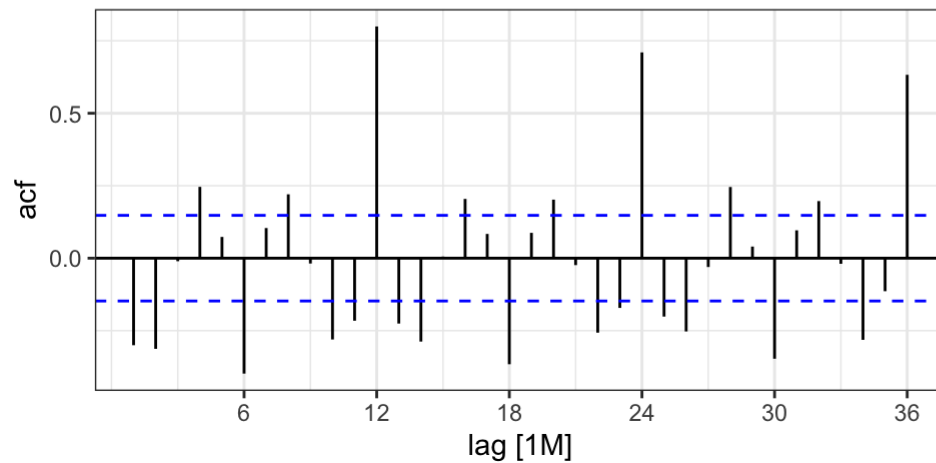
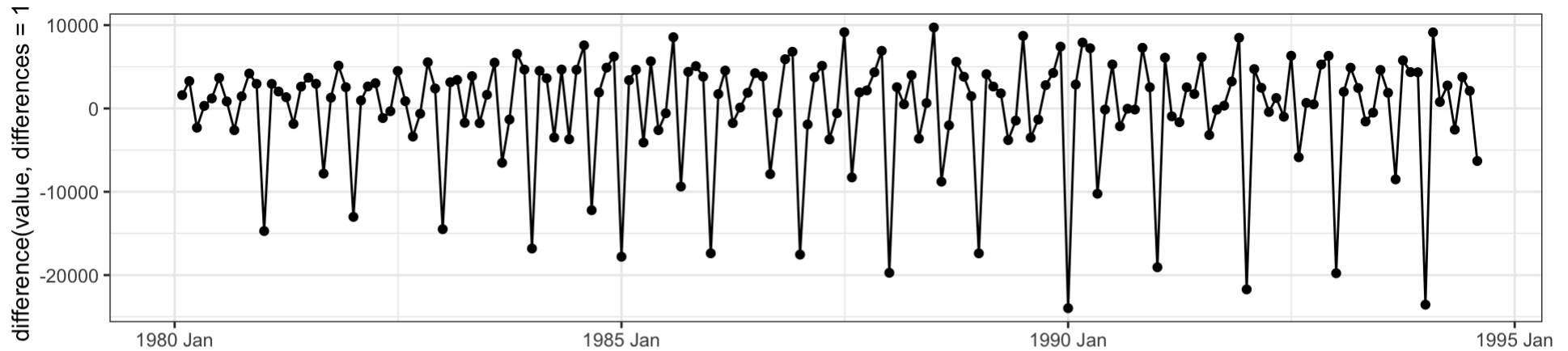
Seasonal plot

```
1 feasts::gg_tsdisplay(wineind, y=value, lag_max = 36)
```



Differencing

```
1 feasts::gg_tsdisplay(wineind, y=difference(value, differences = 1), lag_max = 36)
```



Seasonal ARIMA

We can extend the existing ARIMA model to handle these higher order lags (without including all of the intervening lags).

Seasonal ARIMA $(p, d, q) \times (P, D, Q)_s$:

$$\Phi_P(L^s) \phi_p(L) \Delta_s^D \Delta^d y_t = \delta + \Theta_Q(L^s) \theta_q(L) w_t$$

where

$$\phi_p(L) = 1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_p L^p$$

$$\theta_q(L) = 1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_q L^q$$

$$\Delta^d = (1 - L)^d$$

$$\Phi_P(L^s) = 1 - \Phi_1 L^s - \Phi_2 L^{2s} - \dots - \Phi_P L^{Ps}$$

$$\Theta_Q(L^s) = 1 + \Theta_1 L^s + \Theta_2 L^{2s} + \dots + \Theta_Q L^{Qs}$$

$$\Delta_s^D = (1 - L^s)^D$$

Seasonal Arima - Diff

Lets consider an $ARIMA(0, 0, 0) \times (0, 1, 0)_{12}$:

$$(1 - L^{12}) y_t = \delta + w_t$$

$$y_t = y_{t-12} + \delta + w_t$$

```
1 m = model(  
2   wineind,  
3   m1 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,0, period=12))  
4 )  
5 report(m)
```

Series: value

Model: ARIMA(0,0,0)(0,1,0)[12] w/ drift

Coefficients:

constant

355.0122

s.e. 208.5520

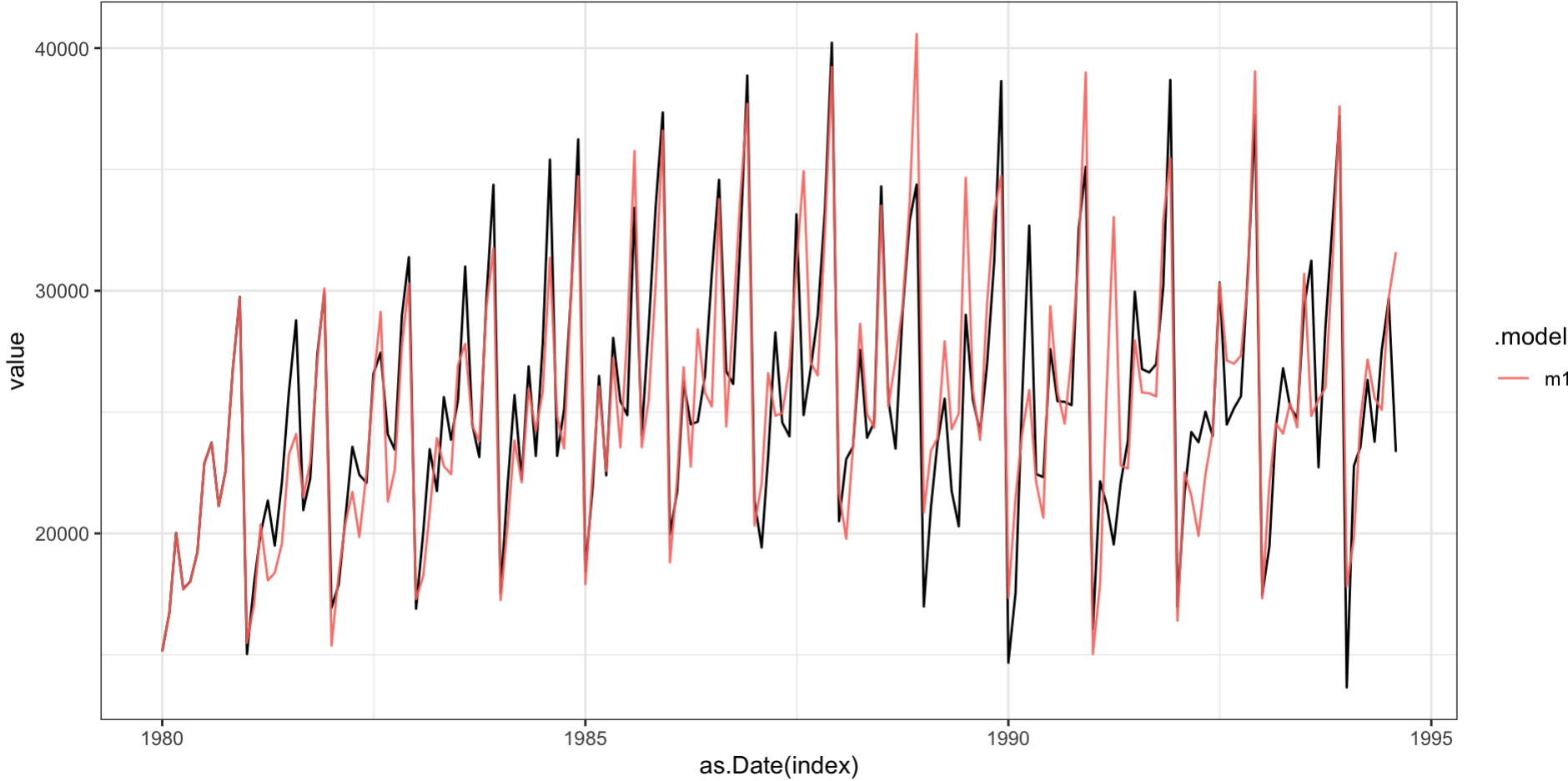
sigma^2 estimated as 7176802: log likelihood=-1526.69

AIC=3057.37 AICc=3057.45 BIC=3063.57

Fitted - Model 1

Fitted

Obs vs fitted

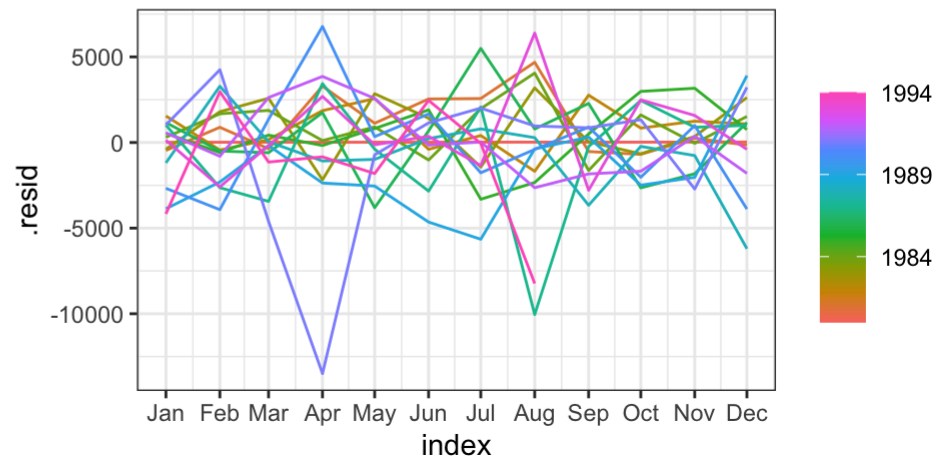
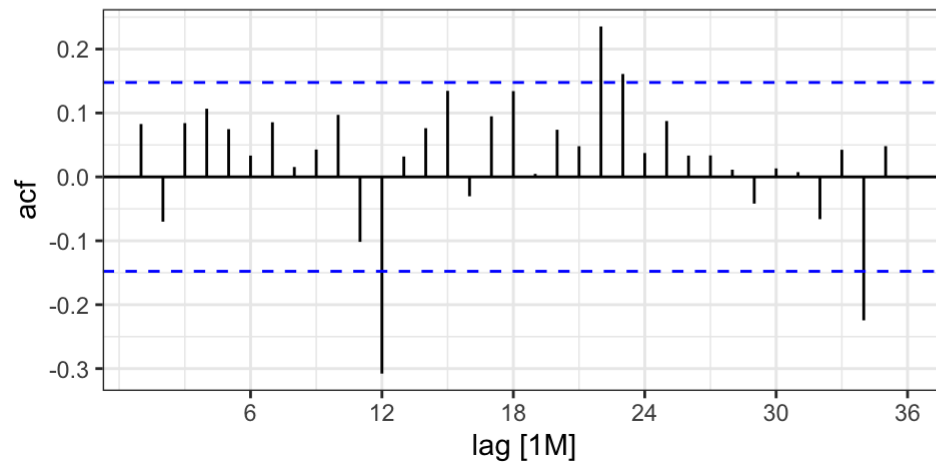
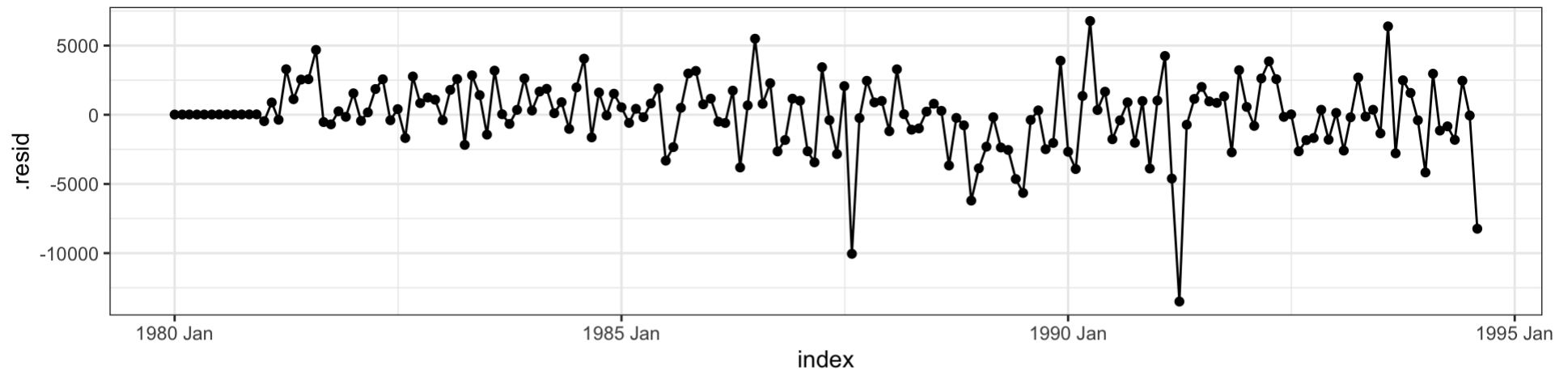


Residuals

Seasonal plot

pACF plot

```
1 residuals(m) |>  
2 feasts::gg_tsdisplay(y=.resid, lag_max=36)
```



Model 2

ARIMA(0, 0, 0) × (0, 1, 1)₁₂:

$$(1 - L^{12})y_t = \delta + (1 + \Theta_1 L^{12})w_t$$
$$y_t = \delta + y_{t-12} + w_t + \Theta_1 w_{t-12}$$

```
1 m = model(  
2   wineind,  
3   m1 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,0, period=12)),  
4   m2 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,1, period=12)),  
5 )  
6 glance(m)
```

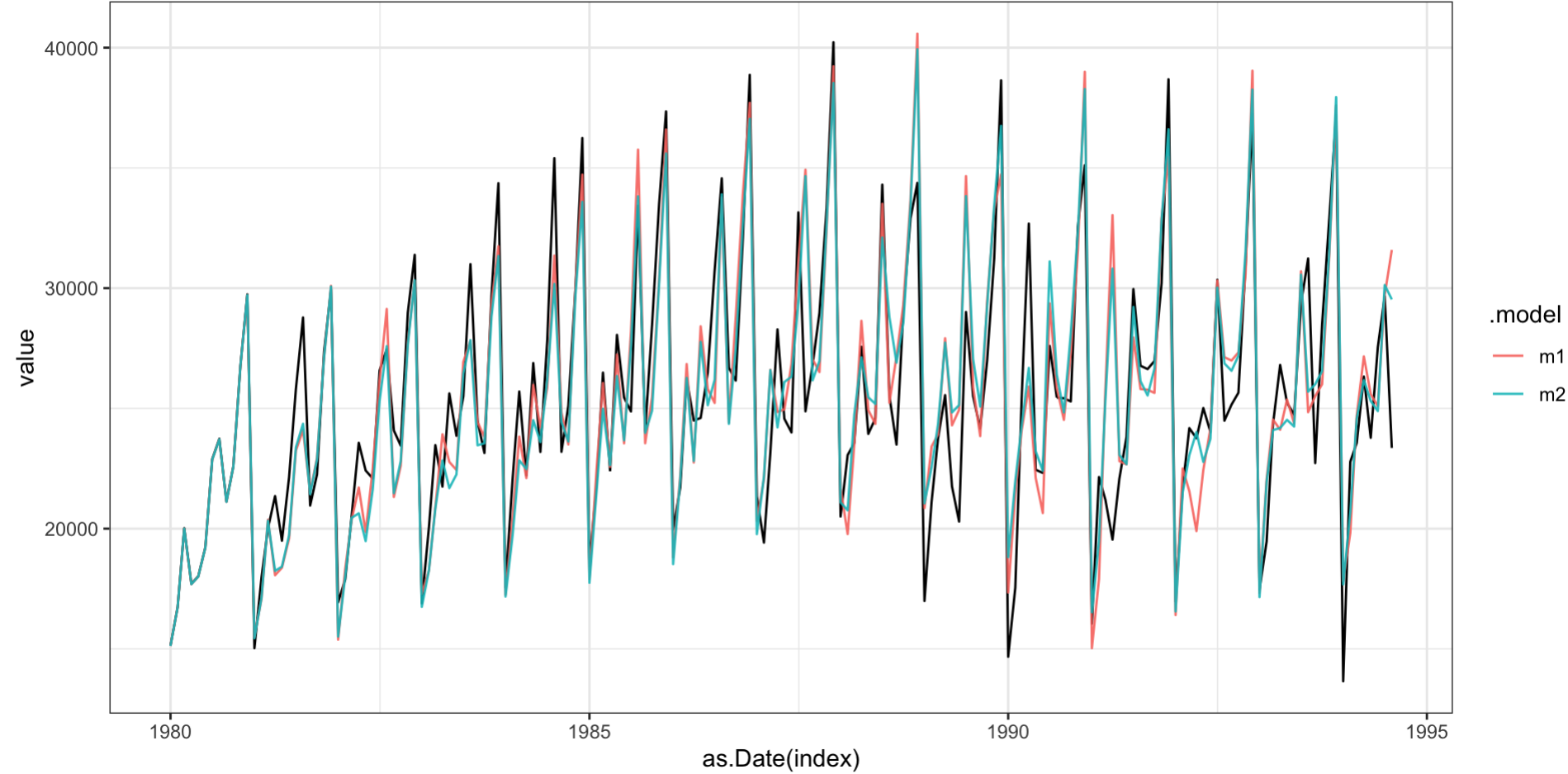
A tibble: 2 × 8

	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	m1	7176802.	-1527.	3057.	3057.	3064.	<cpl [0]>	<cpl [0]>
2	m2	6369103.	-1517.	3041.	3041.	3050.	<cpl [0]>	<cpl [12]>

Fitted - Model 2

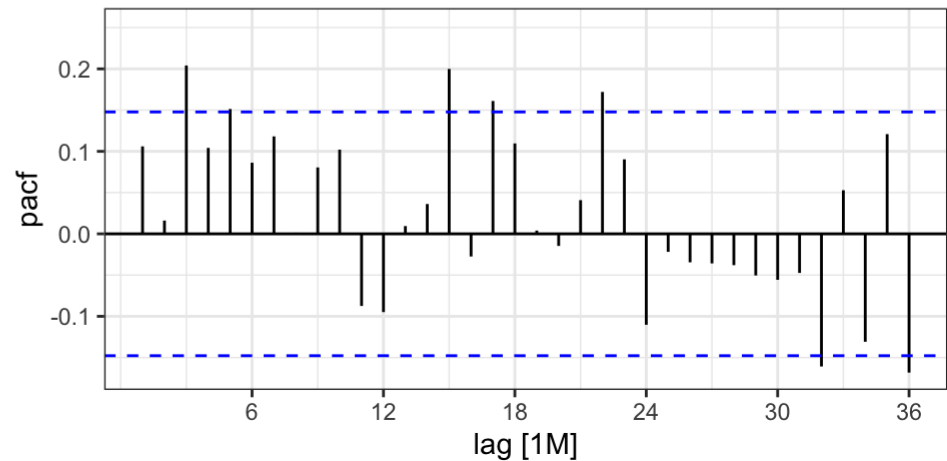
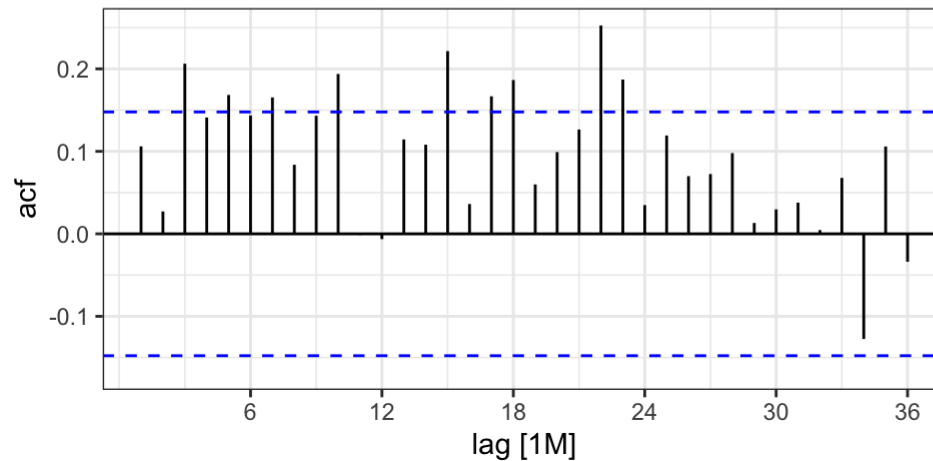
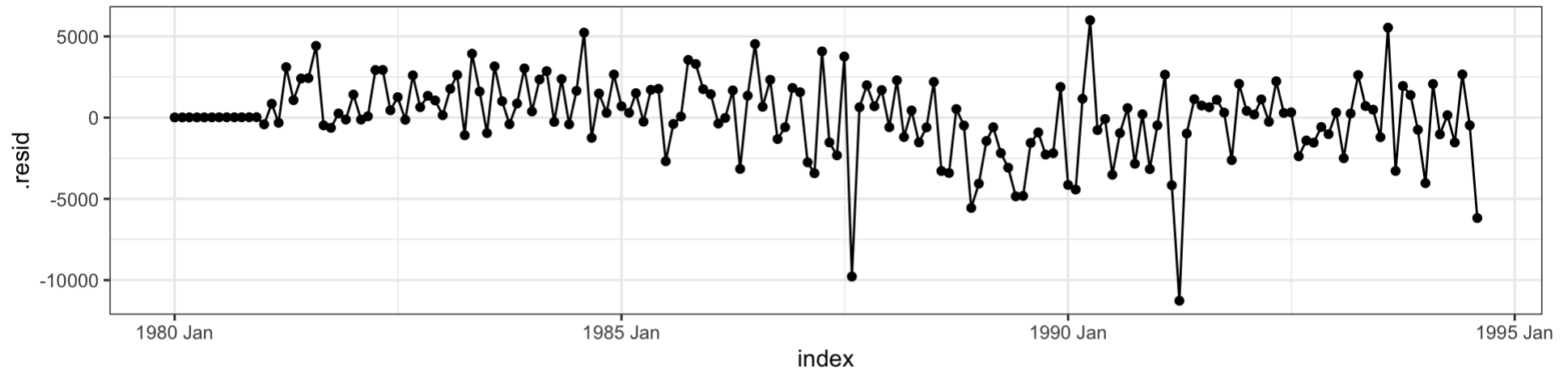
Fitted

Obs vs fitted



Residuals

```
1 residuals(m) |>  
2   filter(.model == "m2") |>  
3   feasts::gg_tsdisplay(y=.resid, plot_type = "partial", lag_max=36)
```



Model 3

ARIMA(0, 1, 0) $\times$ (0, 1, 1)₁₂

$$(1 - L)(1 - L^{12})y_t = \delta + (1 + \Theta_1 L)w_t$$
$$y_t = \delta + y_{t-1} + y_{t-12} - y_{t-13} + w_t + w_{t-12}$$

```
1 m = model(  
2   wineind,  
3   m1 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,0, period=12)),  
4   m2 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,1, period=12)),  
5   m3 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,1, period=12))  
6 )  
7 glance(m)
```

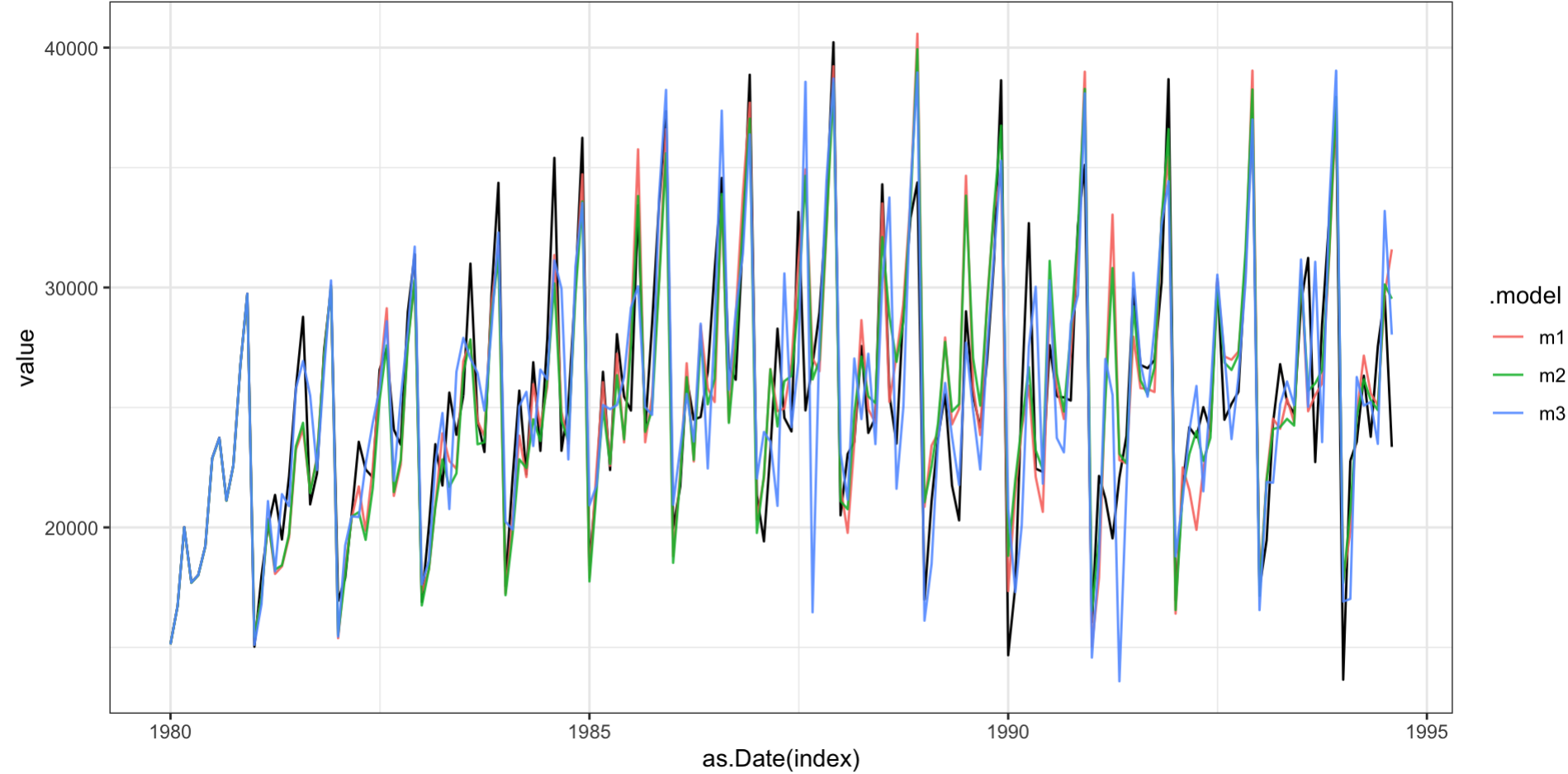
A tibble: 3 × 8

	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	m1	7176802.	-1527.	3057.	3057.	3064.	<cpl [0]>	<cpl [0]>
2	m2	6369103.	-1517.	3041.	3041.	3050.	<cpl [0]>	<cpl [12]>
3	m3	10751355.	-1553.	3109.	3109.	3115.	<cpl [0]>	<cpl [12]>

Fitted model

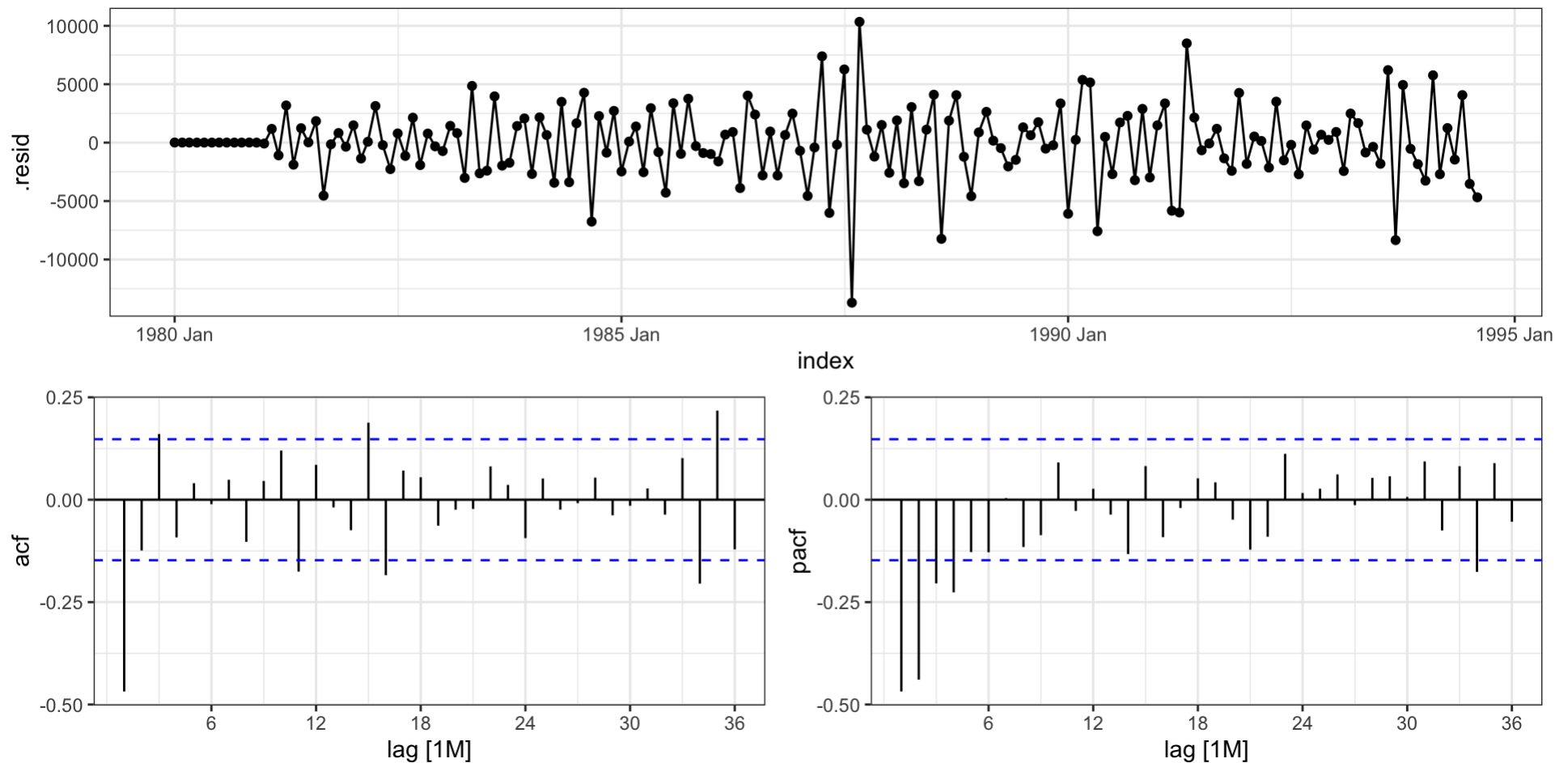
Fitted

Obs vs fitted



Residuals

```
1 residuals(m) |>  
2   filter(.model == "m3") |>  
3   feasts::gg_tsdisplay(y=.resid, plot_type = "partial", lag_max=36)
```



Model 4

ARIMA(1, 1, 0) × (0, 1, 1)₁₂

$$(1 - \phi_1 L)(1 - L)(1 - L^{12})y_t = \delta + (1 + \Theta_1 L)w_t$$

$$y_t = \delta + (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + y_{t-12} - (1 + \phi_1)y_{t-13} + \phi_1 y_{t-14} + w_t + w_{t-12}$$

```
1 m = model(  
2   wineind,  
3   m1 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,0, period=12)),  
4   m2 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,1, period=12)),  
5   m3 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,1, period=12)),  
6   m4 = ARIMA(value ~ pdq(1,1,0) + PDQ(0,1,1, period=12))  
7 )  
8 glance(m)
```

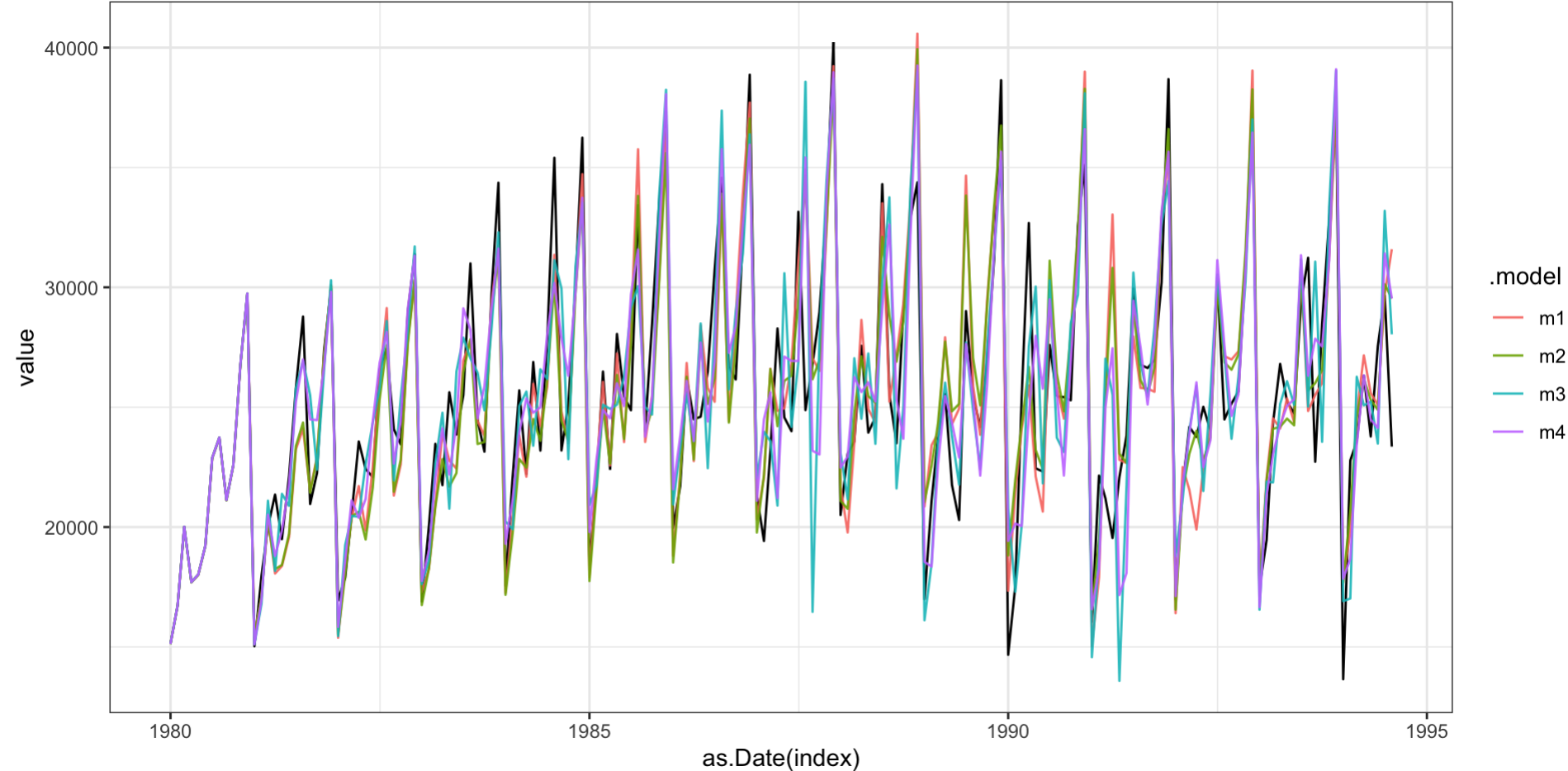
A tibble: 4 × 8

	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	m1	7176802.	-1527.	3057.	3057.	3064.	<cpl [0]>	<cpl [0]>
2	m2	6369103.	-1517.	3041.	3041.	3050.	<cpl [0]>	<cpl [12]>
3	m3	10751355.	-1553.	3109.	3109.	3115.	<cpl [0]>	<cpl [12]>
4	m4	8267920.	-1532.	3070.	3070.	3079.	<cpl [1]>	<cpl [12]>

Fitted model

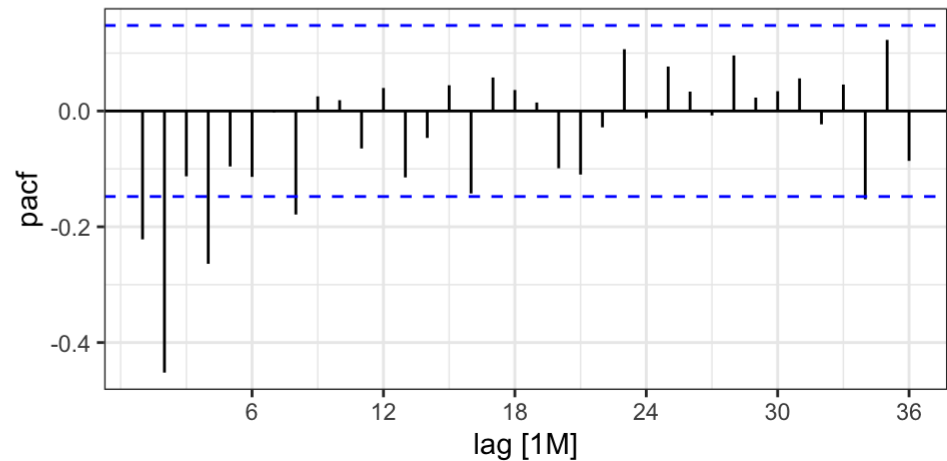
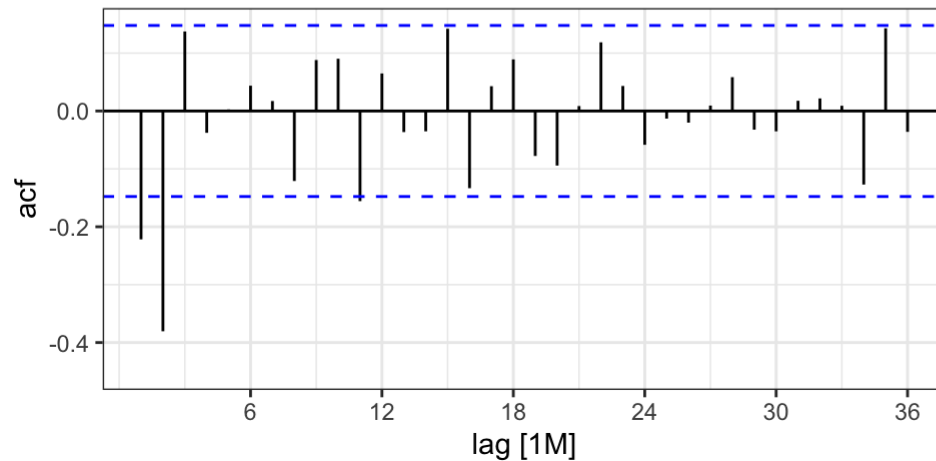
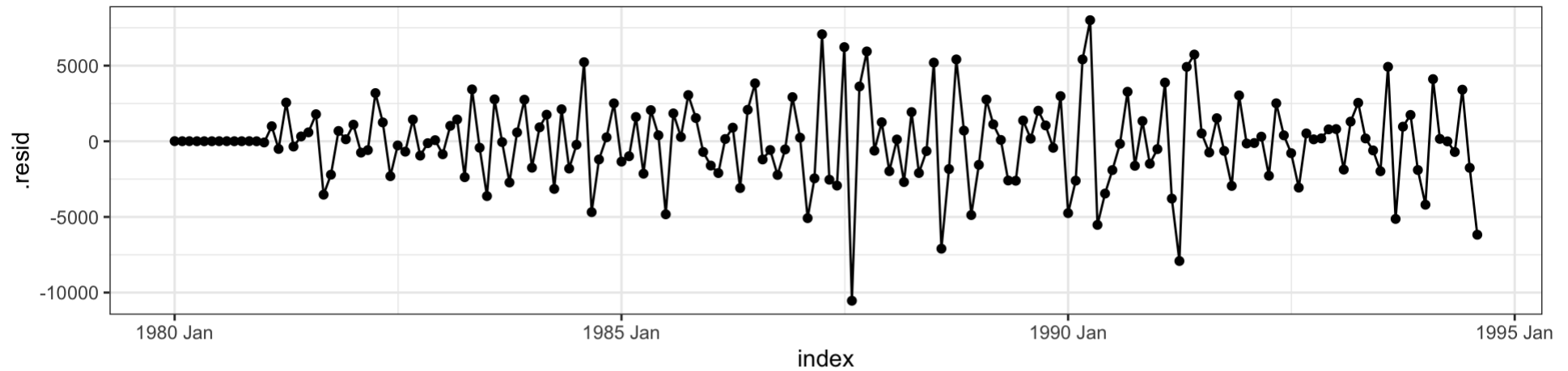
Fitted

Obs vs fitted



Residuals

```
1 residuals(m) |>  
2   filter(.model == "m4") |>  
3   feasts::gg_tsdisplay(y=.resid, plot_type = "partial", lag_max=36)
```



Model 5

ARIMA(1, 1, 2) $\times$ (0, 1, 1)₁₂

$$(1 - \phi_1 L)(1 - L)(1 - L^{12})y_t = \delta + (1 + \theta_1 L + \theta_2 L^2)(1 + \Theta_1 L)w_t$$

$$y_t = \delta + (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + y_{t-12} - (1 + \phi_1)y_{t-13} + \phi_1 y_{t-14} \\ + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_{t-12} + \theta_1 w_{t-13} + \theta_2 w_{t-14}$$

```
1 m = model(wineind,  
2   m1 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,0, period=12)),  
3   m2 = ARIMA(value ~ pdq(0,0,0) + PDQ(0,1,1, period=12)),  
4   m3 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,1, period=12)),  
5   m4 = ARIMA(value ~ pdq(1,1,0) + PDQ(0,1,1, period=12)),  
6   m5 = ARIMA(value ~ pdq(1,1,2) + PDQ(0,1,1, period=12))  
7 ); glance(m)
```

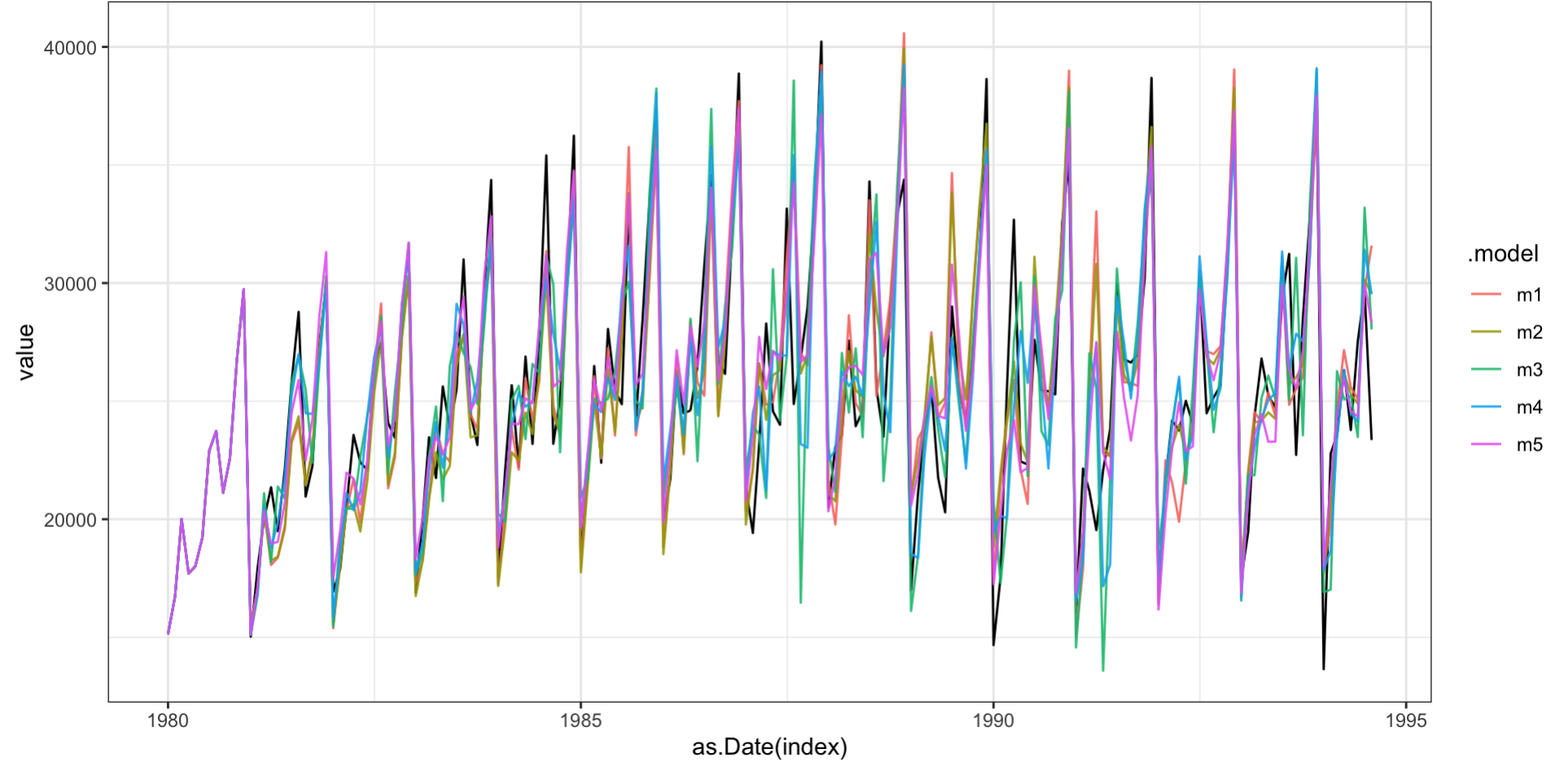
A tibble: 5 × 8

	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	m1	7176802.	-1527.	3057.	3057.	3064.	<dbl [0]>	<dbl [0]>
2	m2	6369103.	-1517.	3041.	3041.	3050.	<dbl [0]>	<dbl [12]>
3	m3	10751355.	-1553.	3109.	3109.	3115.	<dbl [0]>	<dbl [12]>
4	m4	8267920.	-1532.	3070.	3070.	3079.	<dbl [1]>	<dbl [12]>
5	m5	5399312.	-1497.	3004.	3004.	3020.	<dbl [1]>	<dbl [14]>

Fitted model

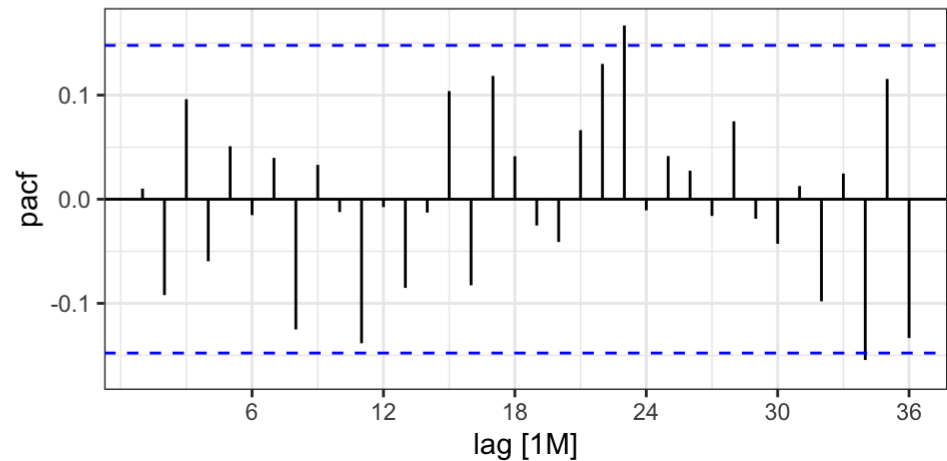
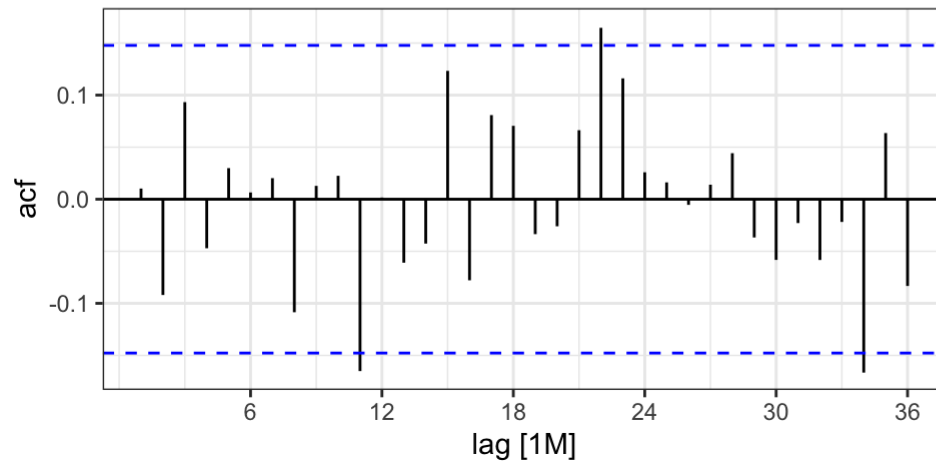
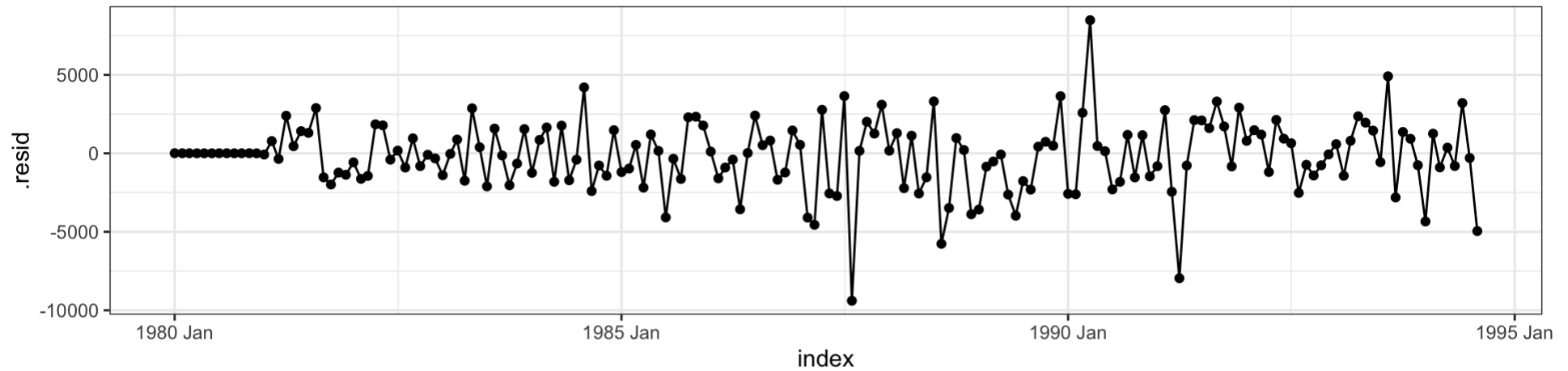
Fitted

Obs vs fitted



Residuals

```
1 residuals(m) |>  
2   filter(.model == "m5") |>  
3   feasts::gg_tsdisplay(y=.resid, plot_type = "partial", lag_max=36)
```



Automated selection

```
1 model(  
2   wineind, ARIMA(value)  
3 ) |>  
4   report()
```

Series: value

Model: ARIMA(1,1,2)(0,1,1)[12]

Coefficients:

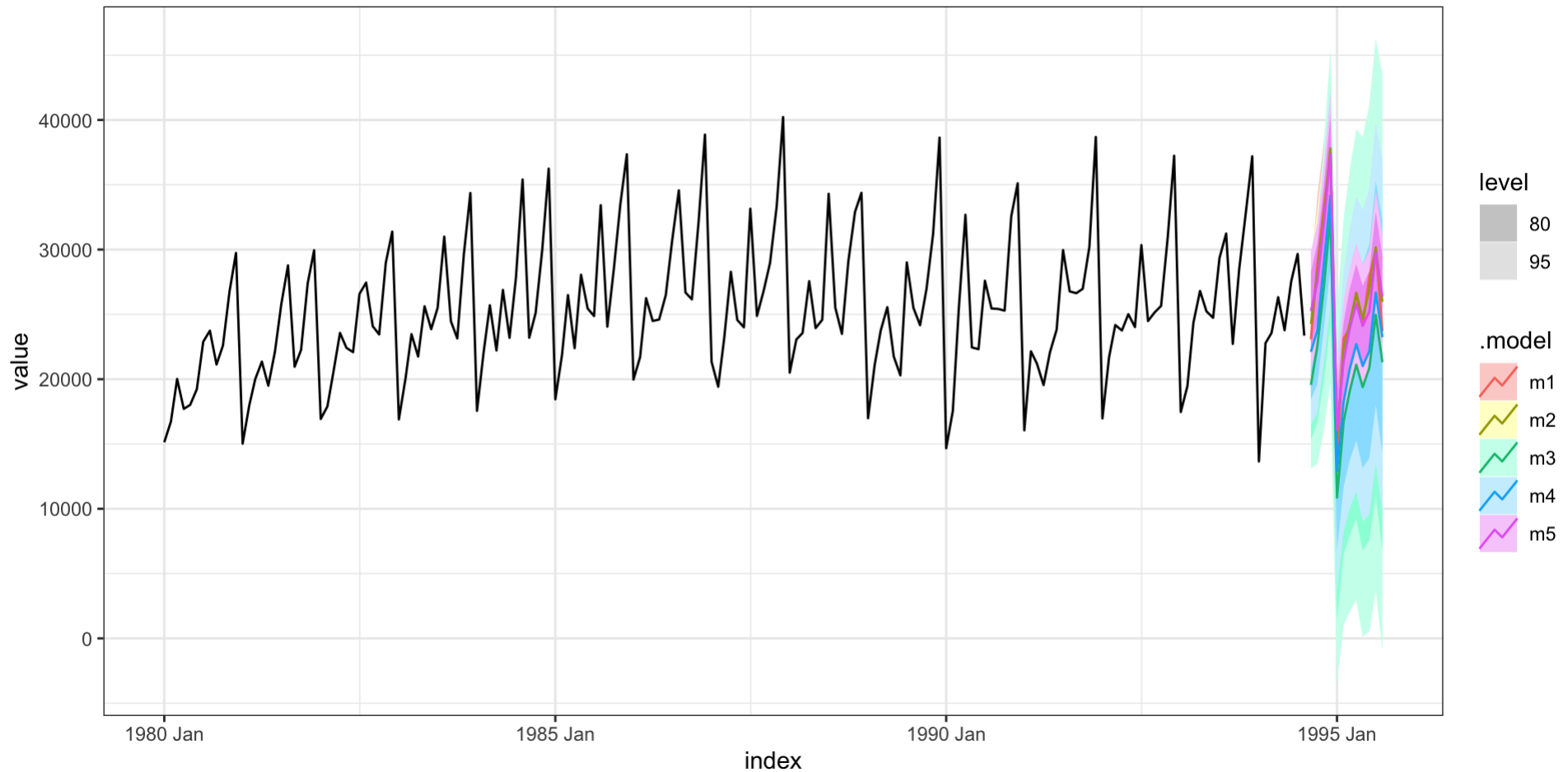
	ar1	ma1	ma2	sma1
	0.4299	-1.4673	0.5339	-0.6600
s.e.	0.2984	0.2658	0.2340	0.0799

sigma^2 estimated as 5399312: log likelihood=-1497.05

AIC=3004.1 AICc=3004.48 BIC=3019.57

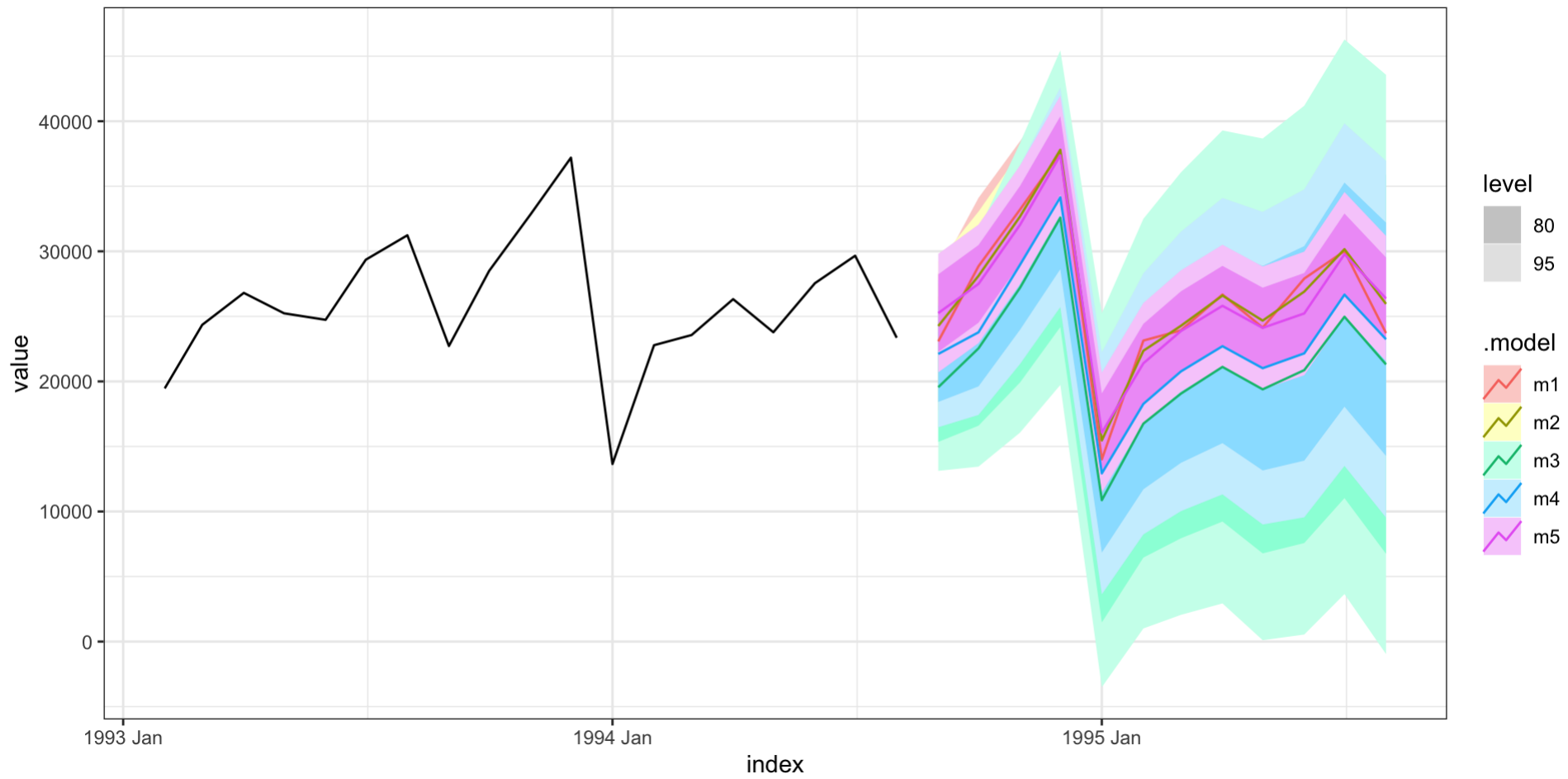
Forecasts

```
1 m |>  
2   forecast(h=12) |>  
3   autoplot(wineind)
```



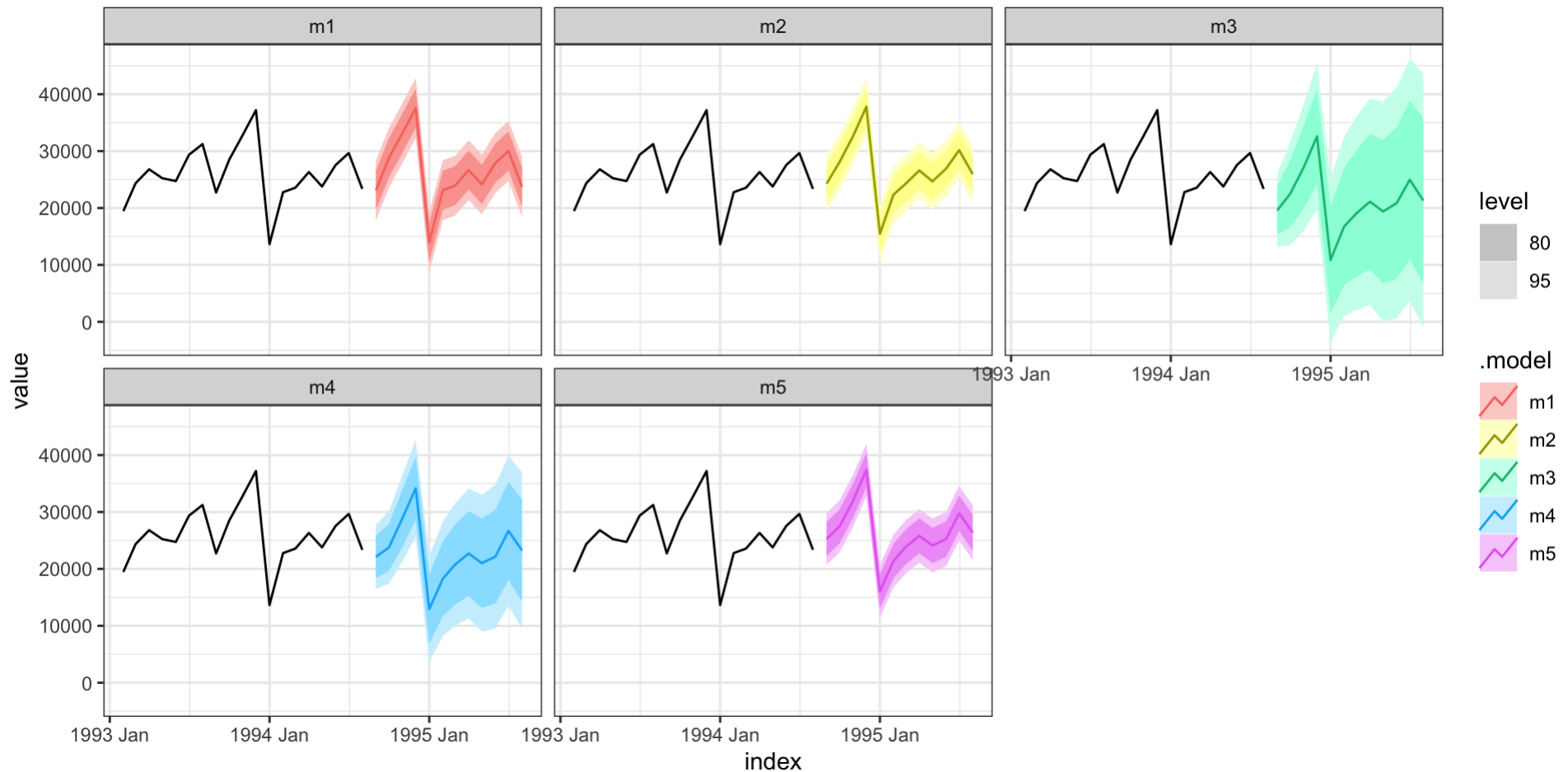
Forecasts (1993-1996)

```
1 m |>
2   forecast(h=12) |>
3   autoplot(
4     wineind |>
5       filter(index > make_yearmonth(1993, 1))
6   )
```



Forecasts

```
1 m |>  
2   forecast(h=12) |>  
3   autoplot(wineind |> filter(index > make_yearmonth(1993, 1))) +  
4   facet_wrap(~.model)
```

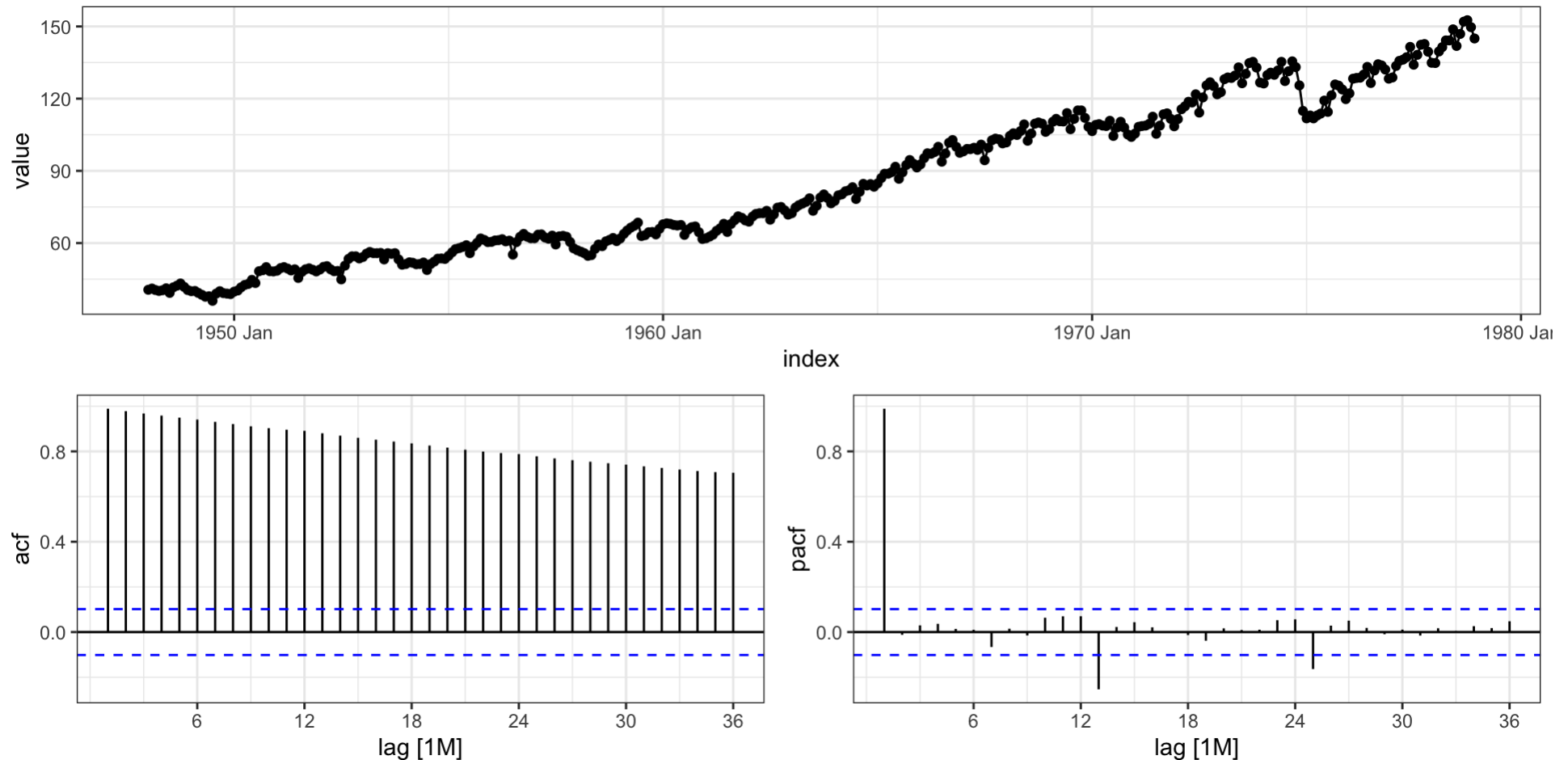


Federal Reserve Board Production Index

prodn from the astsa package

Monthly Federal Reserve Board Production Index (1948-1978)

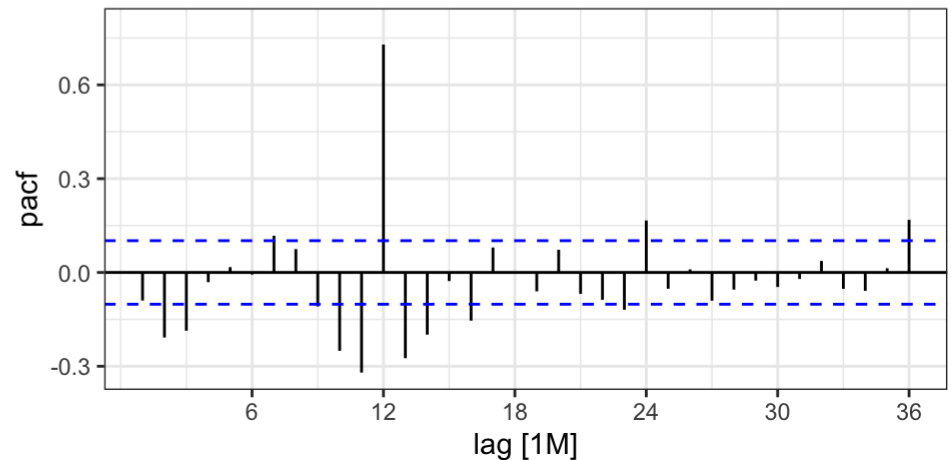
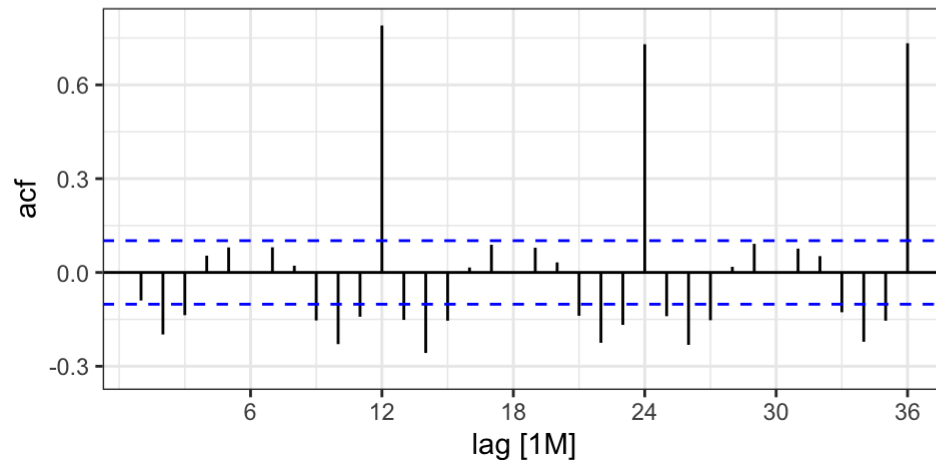
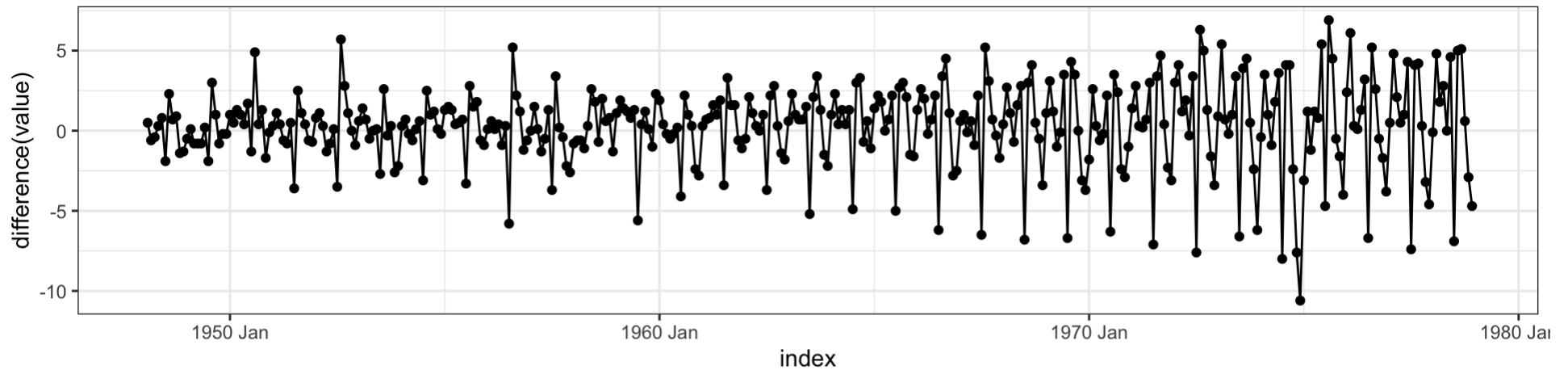
```
1 feasts::gg_tsdisplay(prodn, y=value, plot_type = "partial", lag_max=36)
```



Differencing

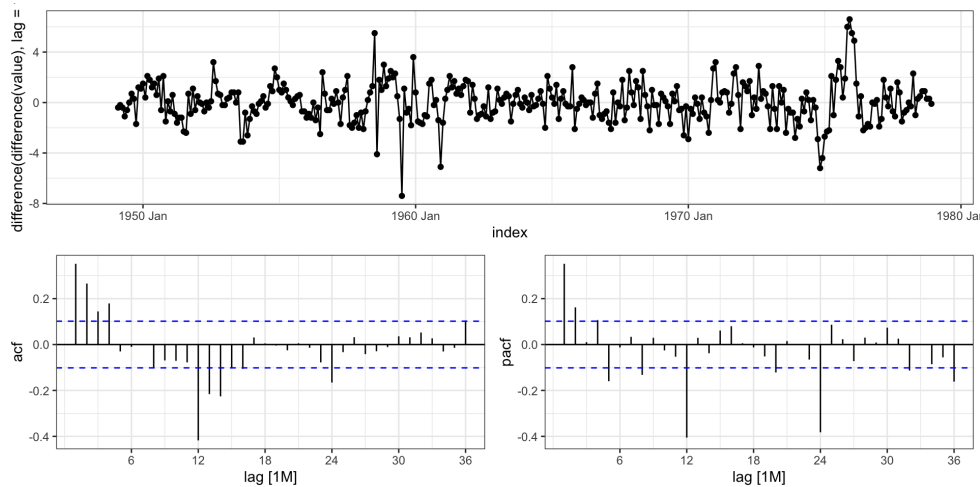
Based on the ACF it seems like standard differencing may be required

```
1 feasts::gg_tsdisplay(prodn, y=difference(value), plot_type = "partial", lag_max=36)
```

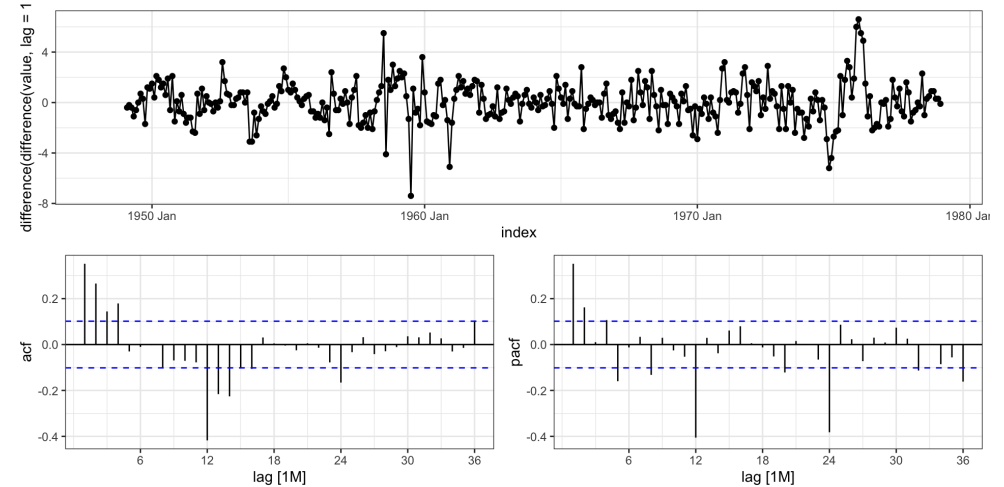


Differencing + Seasonal Differencing

```
1 feasts::gg_tsdisplay(  
2   prodn,  
3   y = difference(value) |>  
4     difference(lag=12),  
5   plot_type = "partial", lag_max=36  
6 )
```



```
1 feasts::gg_tsdisplay(  
2   prodn,  
3   y = difference(value, lag=12) |>  
4     difference(),  
5   plot_type = "partial", lag_max=36  
6 )
```



Model 1

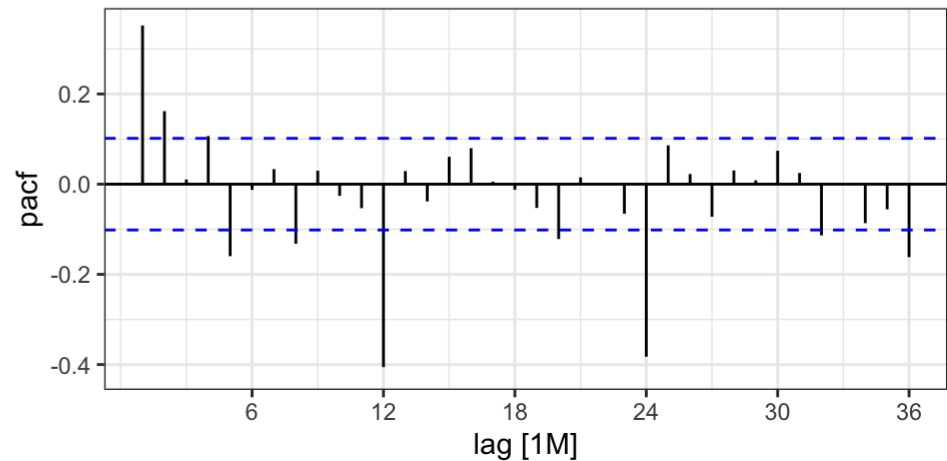
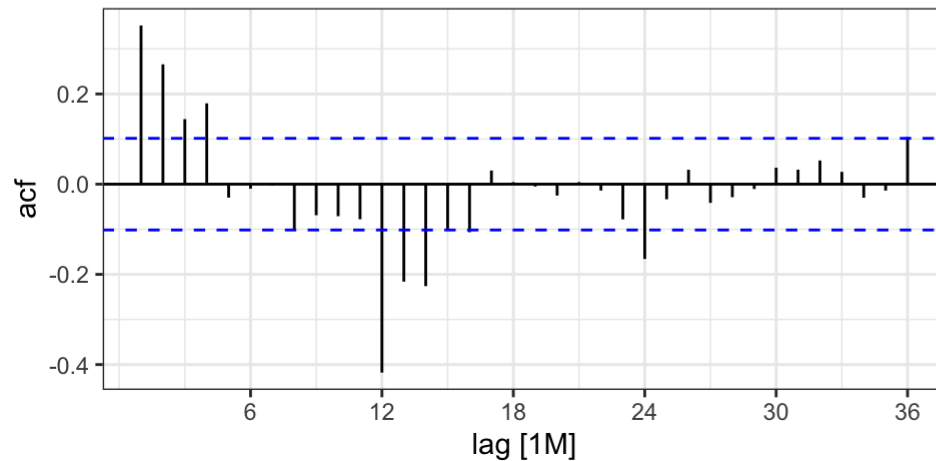
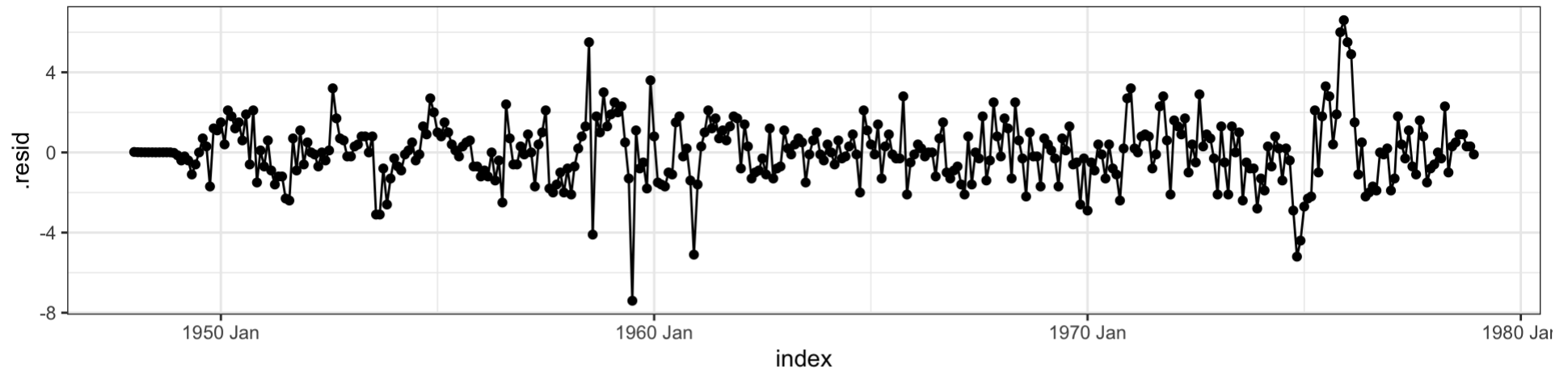
```
1 fr_m = model(  
2   prodn,  
3   m1 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,0, period=12)),  
4 )  
5  
6 glance(fr_m)
```

```
# A tibble: 1 × 8
```

	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	m1	2.52	-675.	1353.	1353.	1356.	<cpl [0]>	<cpl [0]>

Residuals

```
1 residuals(fr_m) |>  
2   filter(.model == "m1") |>  
3   feasts::gg_tsdisplay(y=.resid, plot_type = "partial", lag_max=36)
```



Model 2 - Seasonal MA

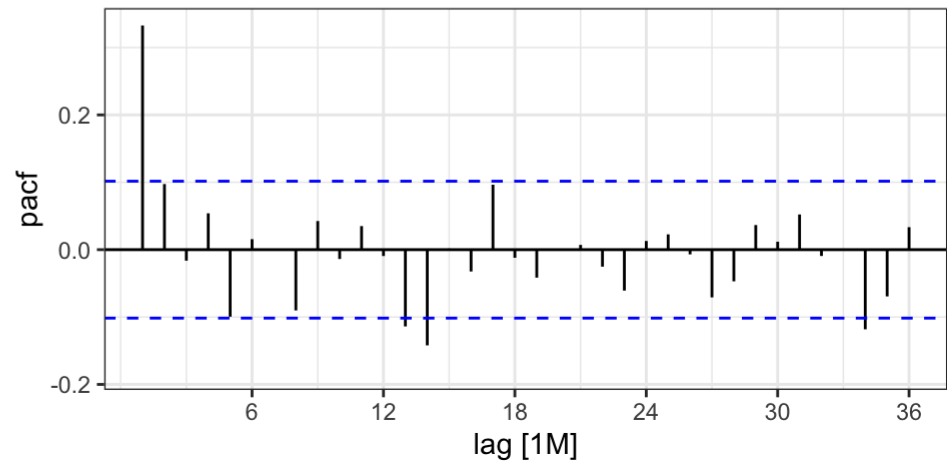
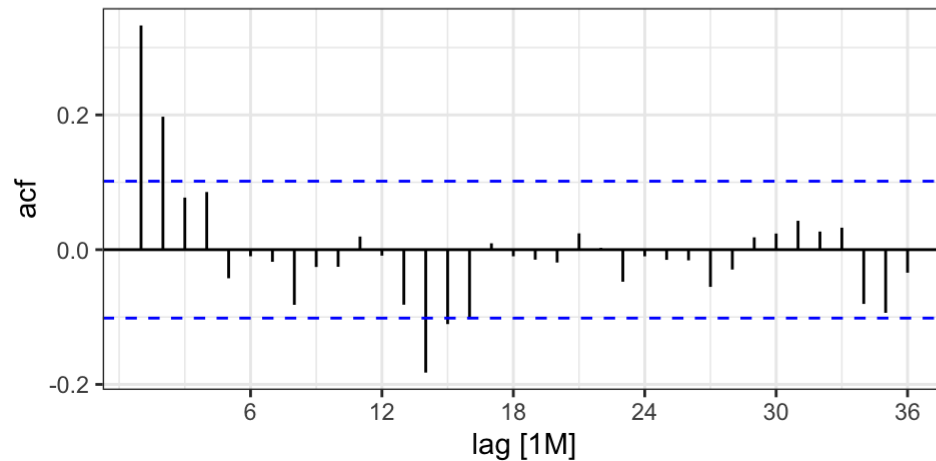
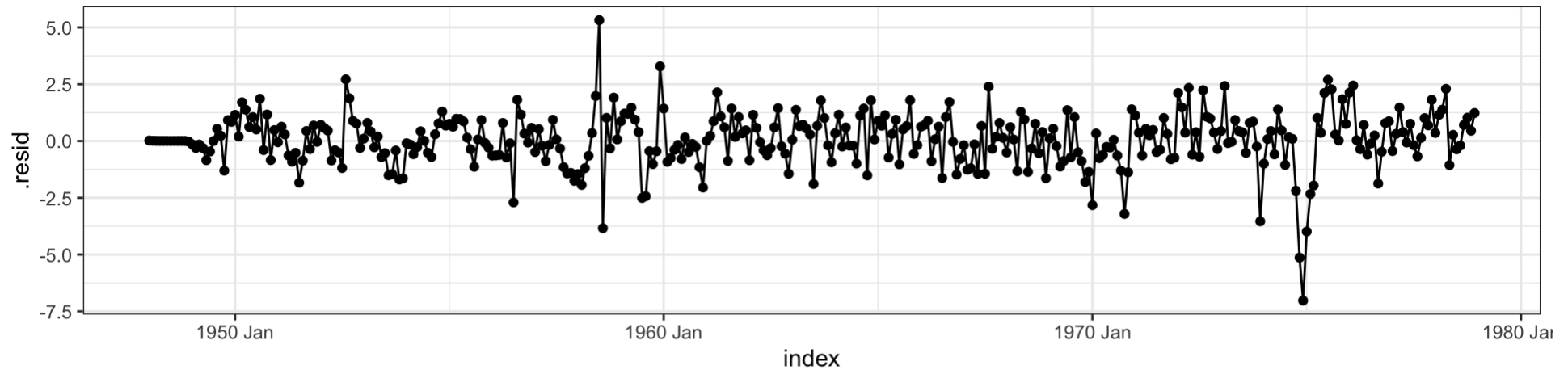
```
1 fr_m = model(  
2   prodn,  
3   m1 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,0, period=12)),  
4   m2_1 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,1, period=12)),  
5   m2_2 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,2, period=12)),  
6   m2_3 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,3, period=12)),  
7   m2_4 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,4, period=12))  
8 )  
9  
10 glance(fr_m)
```

A tibble: 5 × 8

	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	m1	2.52	-675.	1353.	1353.	1356.	<cpl [0]>	<cpl [0]>
2	m2_1	1.62	-599.	1203.	1203.	1210.	<cpl [0]>	<cpl [12]>
3	m2_2	1.62	-599.	1204.	1204.	1216.	<cpl [0]>	<cpl [24]>
4	m2_3	1.51	-588.	1183.	1183.	1199.	<cpl [0]>	<cpl [36]>
5	m2_4	1.51	-587.	1185.	1185.	1204.	<cpl [0]>	<cpl [48]>

Residuals - Model 2-3

```
1 residuals(fr_m) |>  
2   filter(.model == "m2_3") |>  
3   feasts::gg_tsdisplay(y=.resid, plot_type = "partial", lag_max=36)
```



Model 3 - Adding AR

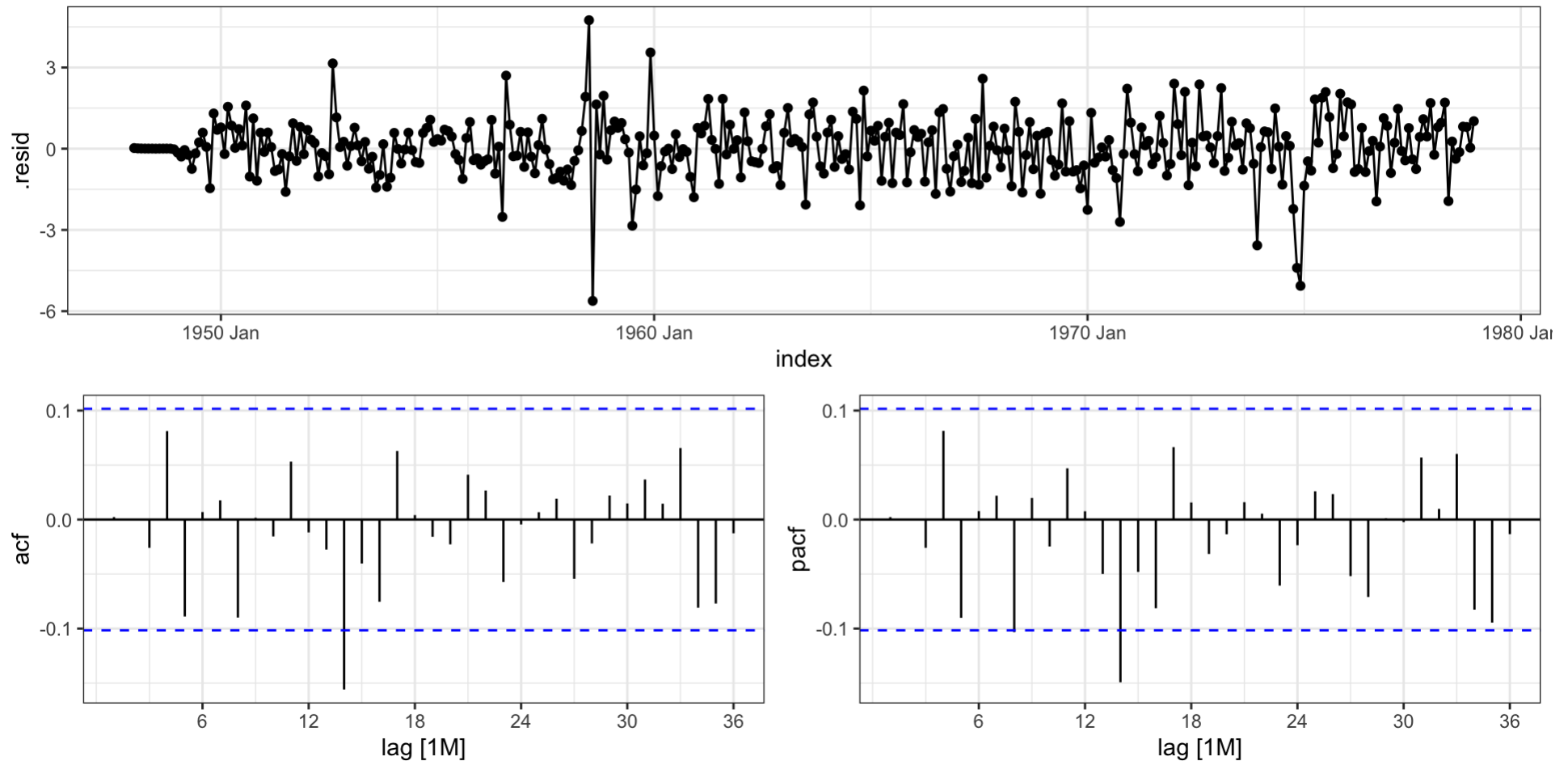
```
1 fr_m = model(  
2   prodn,  
3   m1 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,0, period=12)),  
4   m2_3 = ARIMA(value ~ pdq(0,1,0) + PDQ(0,1,3, period=12)),  
5   m3_1 = ARIMA(value ~ pdq(1,1,0) + PDQ(0,1,3, period=12)),  
6   m3_2 = ARIMA(value ~ pdq(2,1,0) + PDQ(0,1,3, period=12))  
7 )  
8  
9 glance(fr_m)
```

```
# A tibble: 4 × 8
```

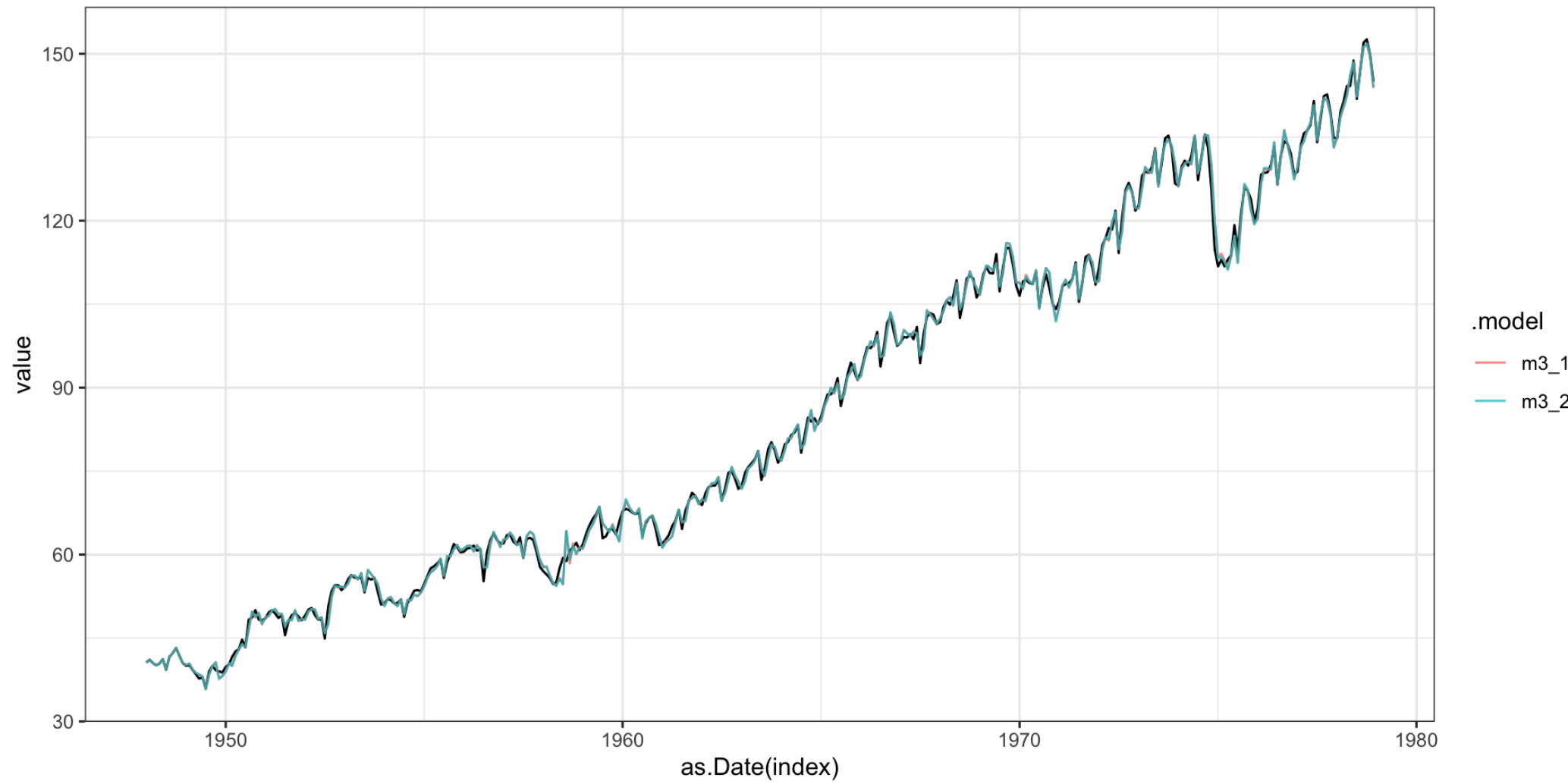
	.model	sigma2	log_lik	AIC	AICc	BIC	ar_roots	ma_roots
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<list>	<list>
1	m1	2.52	-675.	1353.	1353.	1356.	<cpl [0]>	<cpl [0]>
2	m2_3	1.51	-588.	1183.	1183.	1199.	<cpl [0]>	<cpl [36]>
3	m3_1	1.34	-566.	1142.	1142.	1161.	<cpl [1]>	<cpl [36]>
4	m3_2	1.33	-564.	1140.	1140.	1163.	<cpl [2]>	<cpl [36]>

Residuals - Model 3-2

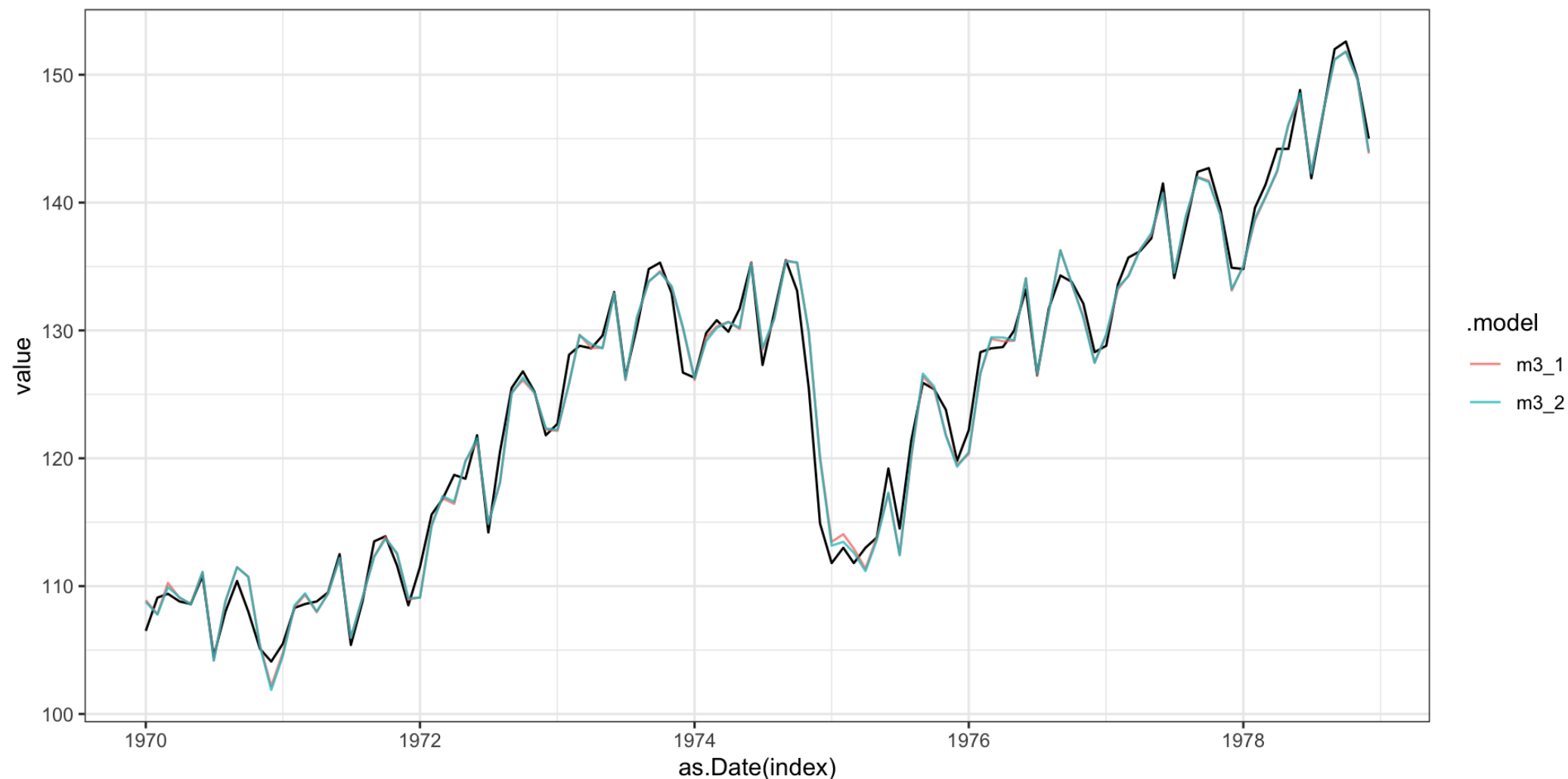
```
1 residuals(fr_m) |>  
2   filter(.model == "m3_2") |>  
3   feasts::gg_tsdisplay(y=.resid, plot_type = "partial", lag_max=36)
```



Model Fit

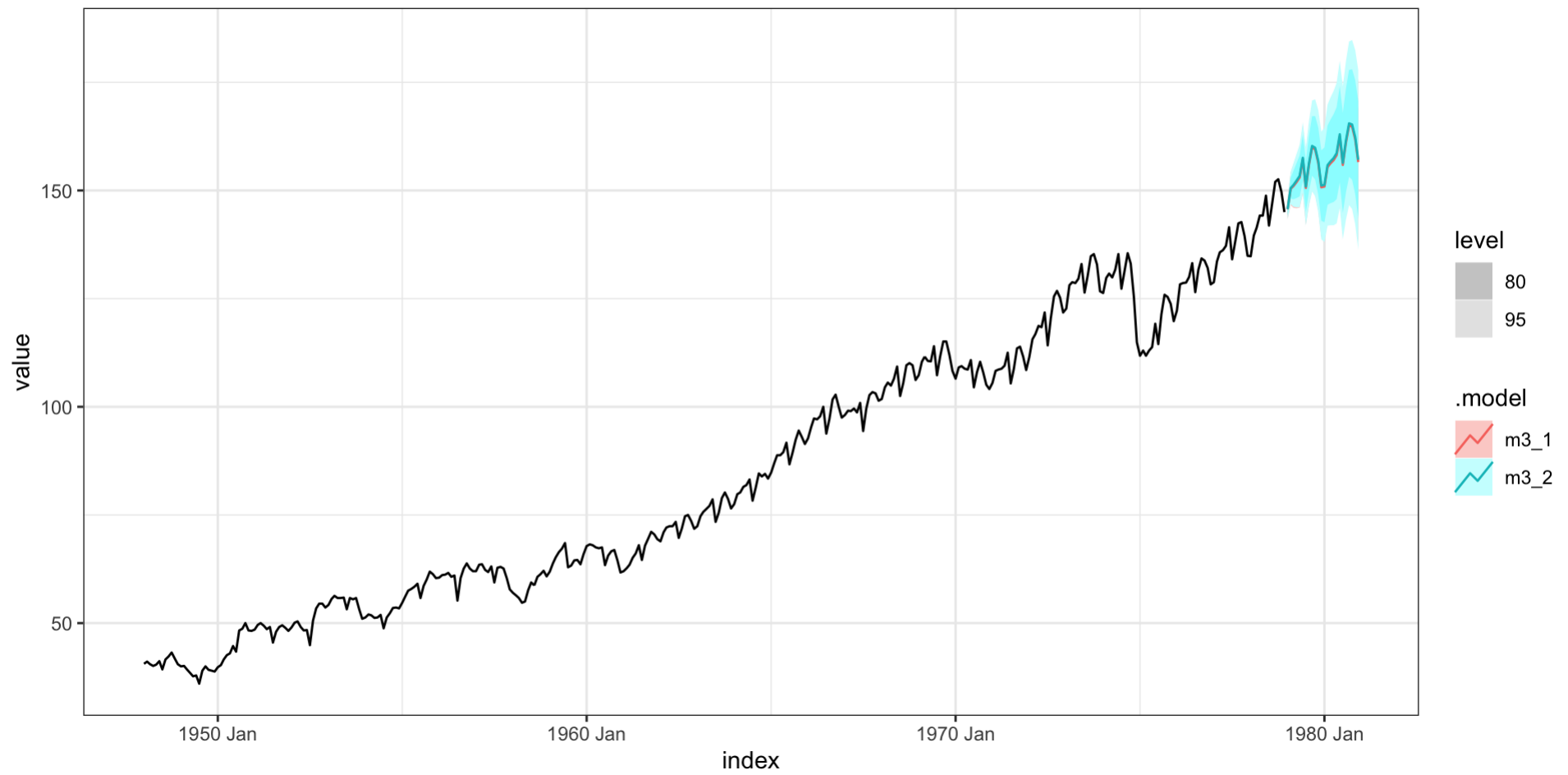


Model Fit (1970-1979)



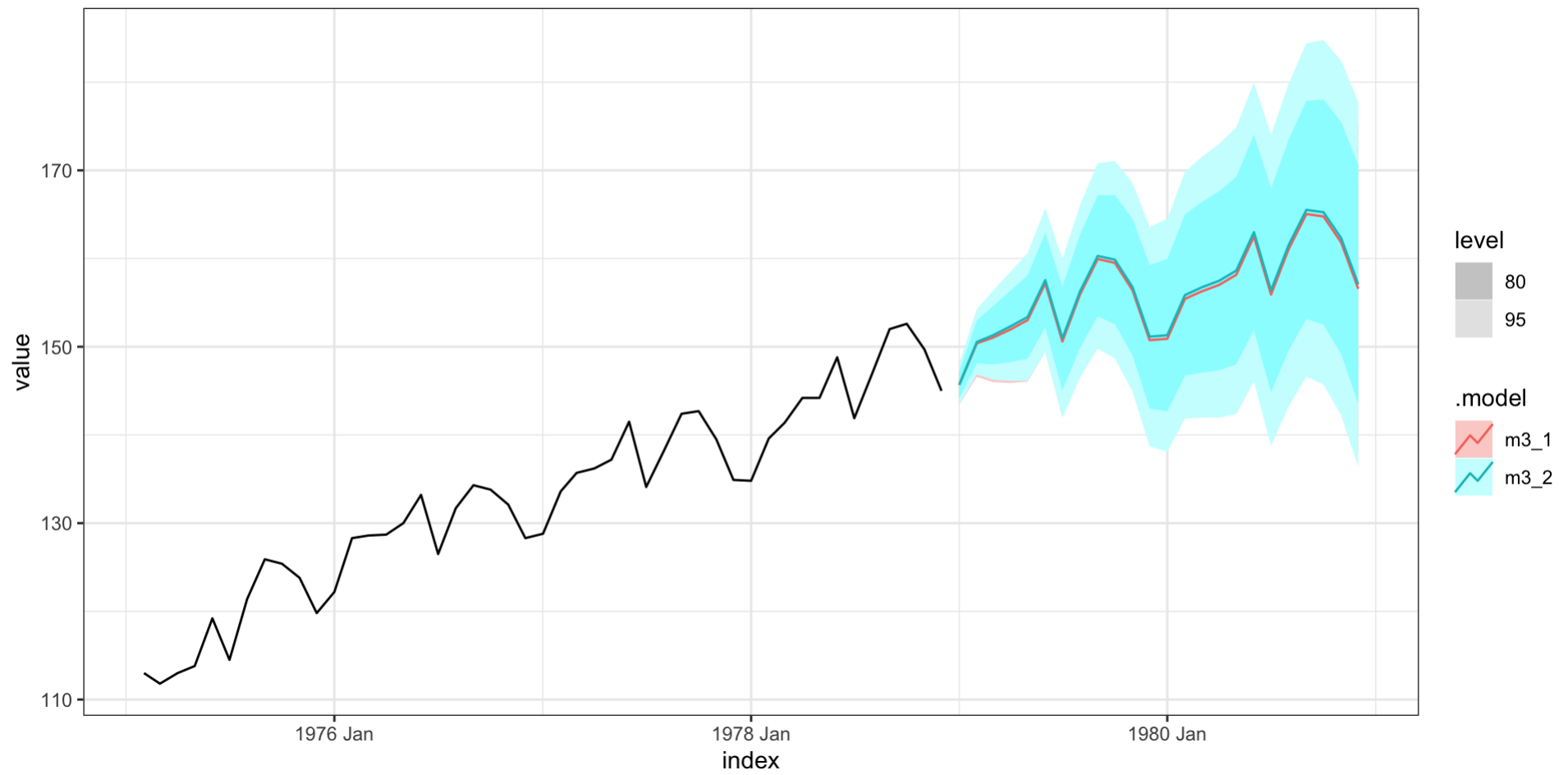
Model Forecast

```
1 fr_m |>
2   forecast(h=24) |>
3   filter(.model %in% c("m3_1", "m3_2")) |>
4   autoplot(
5     prodn
6   )
```



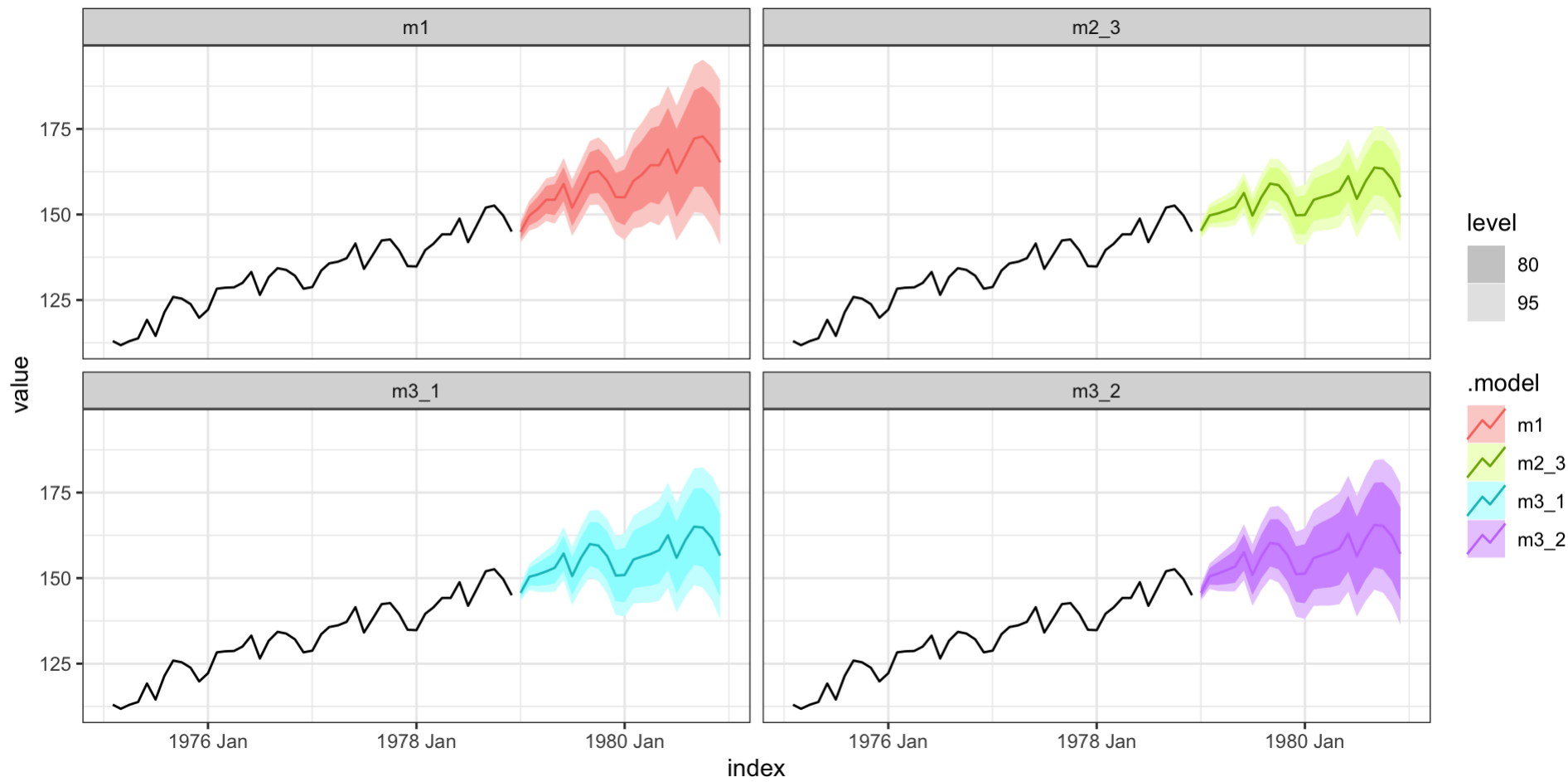
Model Forecast (1975-1981)

```
1 fr_m |>
2   forecast(h=24) |>
3   filter(.model %in% c("m3_1", "m3_2")) |>
4   autoplot(
5     prodn |> filter(index > make_yearmonth(1975, 1))
6   )
```



Model Forecast (comparison)

```
1 fr_m |>
2   forecast(h=24) |>
3   autoplot(
4     prodn |> filter(index > make_yearmonth(1975, 1))
5   ) +
6     facet_wrap(~.model)
```



Auto ARIMA - Model Fit

```
1 model(prodn, ARIMA(value)) |>  
2   report()
```

Series: value

Model: ARIMA(2,0,1)(0,1,1)[12] w/ drift

Coefficients:

	ar1	ar2	ma1	sma1	constant
	1.6696	-0.6944	-0.4166	-0.6684	0.0857
s.e.	0.0999	0.0973	0.1286	0.0352	0.0125

sigma^2 estimated as 1.398: log likelihood=-573.59

AIC=1159.18 AICc=1159.42 BIC=1182.5

Exercise - Cortecosteroid Drug Sales

Monthly cortecosteroid drug sales in Australia from 1992 to 2008.

```
1 data(h02, package="fpp")
2 h02 = as_tsibble(h02)
3
4 feasts::gg_tsdisplay(h02, y=value, plot_type = "partial", lag_max=36)
```

